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VECTORS AND MATRICES

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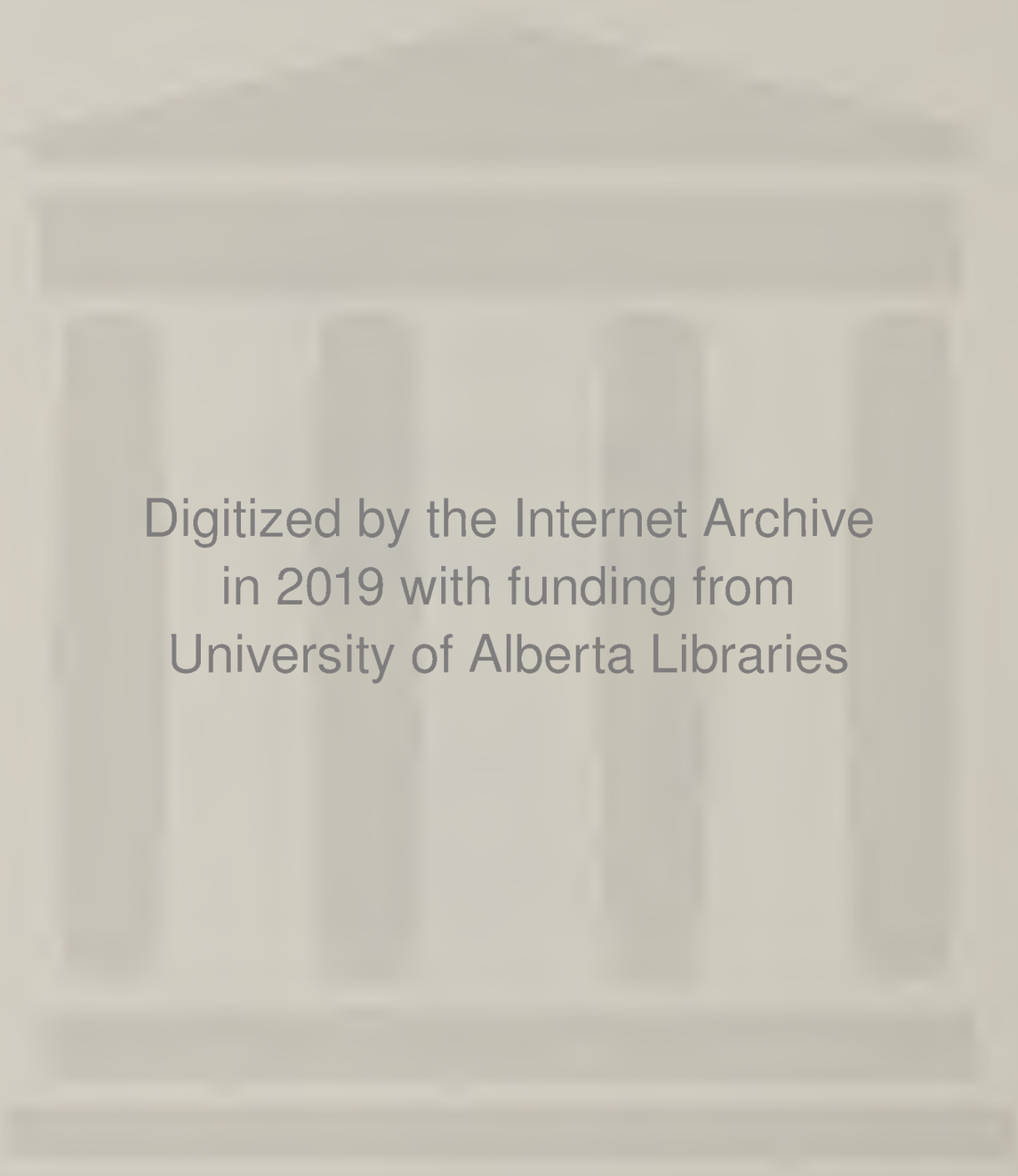
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VECTORS AND MATRICES

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PREFACE

In this text the student is led from a study of physical vectors, in particular of their geometric representatives, to a study of algebraic vectors which provides the student with a precise account of this branch of mathematics and an introduction to the abstract ideas of vector spaces. This work is then applied mainly to the development of a number of ideas in plane co-ordinate geometry (many of which are already familiar to the student) and the extension of these ideas to three-dimensional geometry.

Discussions of systems of linear equations, their solution or lack of a solution, form a significant part of the work in this text. These discussions lead naturally to and culminate in an introduction to the study of matrices.

The authors are indebted to Mr. James Cisarchuk and his team of student assistants from the University of Waterloo for their help in preparing the set of answers for the Exercises.

H. A. E.
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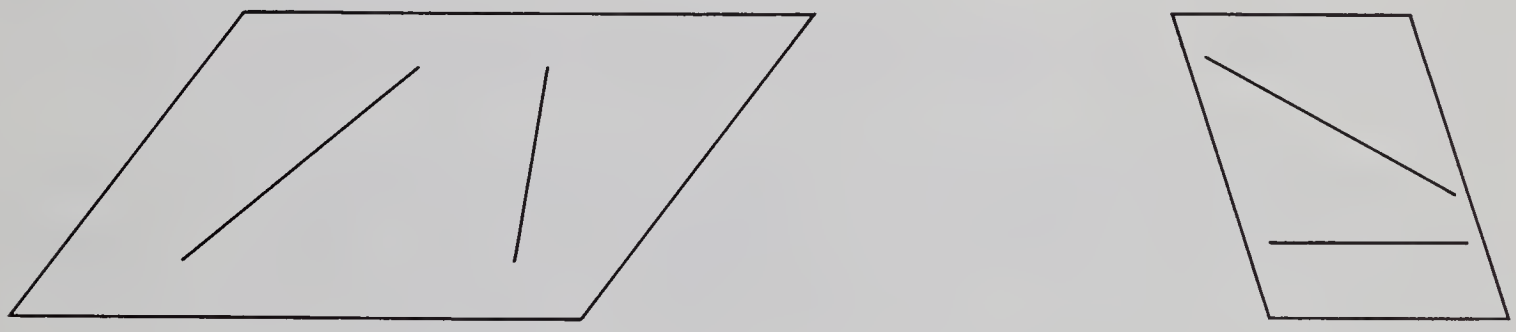
Chapter 1

VECTORS

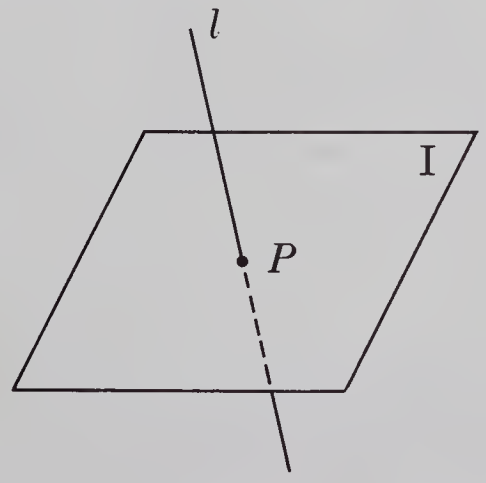
1.1. Lines and Planes in Space

In most of our geometrical work in previous years, we studied lines and other figures in a plane. However, geometrical figures do exist in 3-dimensional space and not just in the 2-dimensional space that we studied previously. It is, of course, a little more difficult to represent 3-dimensional figures on the 2-dimensional surface of a sheet of paper. Certain conventions have been established to enable us to make this representation without ambiguity.

A region of a plane is represented by a parallelogram. Lines and line segments are represented by line segments.



A line can intersect a plane. We show this situation in the diagram below. The plane I is cut by the line l at a point P. The dotted segment of the line indicates where the line is behind the plane from our point of view.



In 3-dimensional geometry, we require certain axioms and definitions in addition to those that we have already used in 2-dimensional geometry.

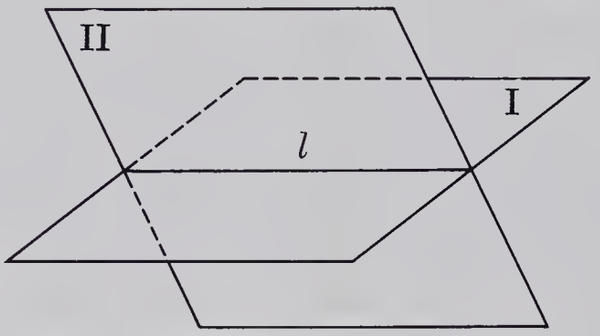
Axiom 1. If two points A and B lie in a plane I, then the line AB determined by these points lies entirely in the plane I; that is, all points on the line are coplanar in I.

Axiom 2. If two distinct planes I and II intersect, they intersect in a line m.

Axiom 3. Through any three distinct noncollinear points, there is one and only one plane.

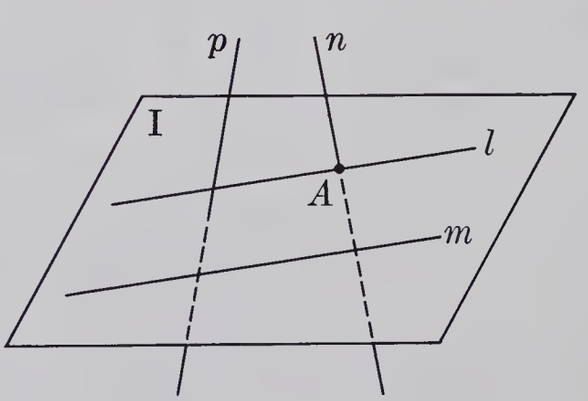
We note, as a consequence of Axiom 1, that a plane I and a line l not lying entirely in the plane can intersect in at most one point. If the line does not intersect the plane, then it is parallel to the plane.

The diagram shows two planes I and II intersecting in a line.



DEFINITION. Skew lines are lines in space which do not intersect and are not parallel.

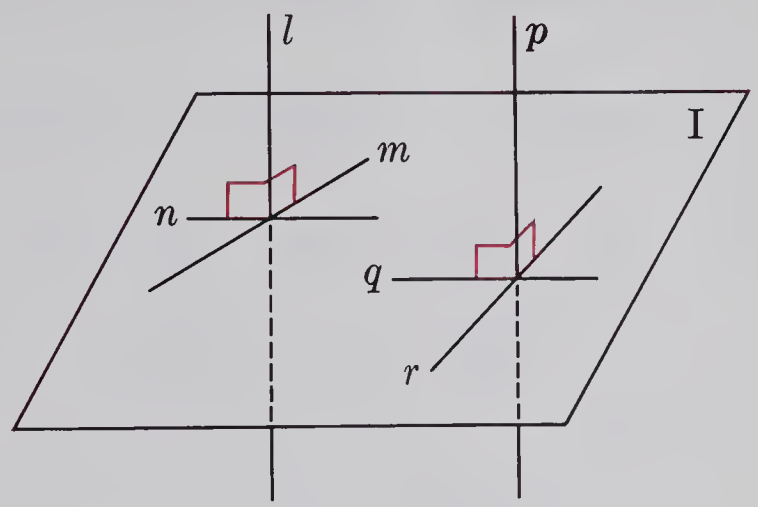
In the diagram below, the lines l and m are coplanar in I and do not intersect. They are, therefore, parallel lines by definition. The lines l and n are not coplanar but do intersect at A. (An enlarged dot is, by convention, used to indicate the point of intersection.) l and n are, therefore, not skew lines.



Line p is not coplanar with l and m, nor does it intersect either l or m. By definition, p and l are skew lines as are p and m.

DEFINITION. A line l is perpendicular to a plane I at a point N if it is perpendicular to every line in the plane passing through N.

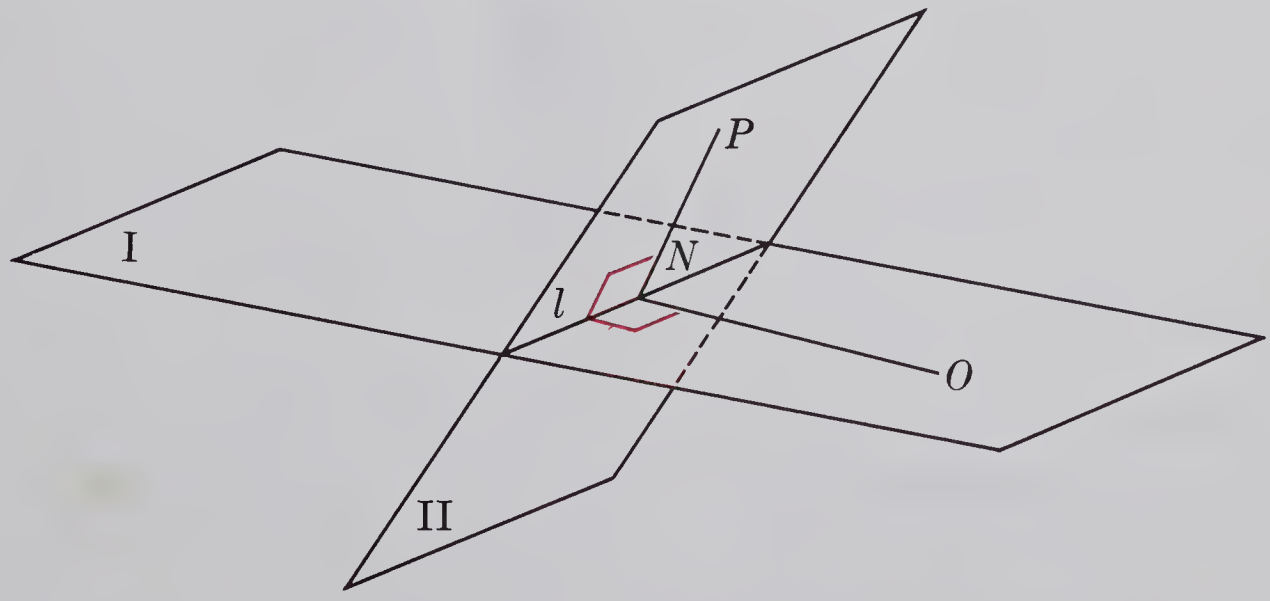
In the diagram, the line l is perpendicular to both m and n which are coplanar in I . l is perpendicular to both m and n and this is indicated by the usual marking for indicating right angles. It can be proved that a line is perpendicular to any line in a plane if it is perpendicular to any two distinct lines in the plane. l is, therefore, perpendicular to I . Similarly, p is perpendicular to I .



In plane geometry we have proved that if two lines are both perpendicular to the same line, then they are parallel to each other. In the same way, it may be proved that if two lines are both perpendicular to the same plane, then they are parallel to each other. In the diagram, $l \parallel p$.

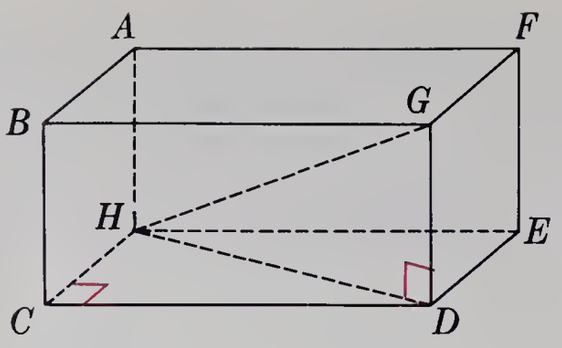
When two planes intersect, they form an angle between the two planes called the *dihedral angle*. How do we measure such an angle?

In the diagram below, N is any point on the line l of intersection of the two planes. ON is perpendicular to l in plane I and PN is perpendicular to l in plane II . $\angle ONP$ can be measured in the usual way and the measure of $\angle ONP$ is defined to be the measure of the dihedral angle.



DEFINITION. The measure of a dihedral angle is the measure of the plane angle formed by the perpendiculars to the line of intersection of the planes at any point on the line of intersection.

Example 1. In the diagram, the figure represents a rectangular prism. If $CD = 8$ in., $DE = 5$ in., and $BC = 6$ in., calculate the length of the diagonal GH and the measure of $\angle GHD$.



Solution: In $\triangle HCD$,

$$\angle HCD = 90^\circ .$$

Therefore,

$$\begin{aligned} HD^2 &= HC^2 + CD^2 \\ &= 25 + 64 . \\ HD &= \sqrt{89} \text{ in.} \end{aligned}$$

In $\triangle GHD$,

$$\angle GDH = 90^\circ .$$

Therefore,

$$\begin{aligned} GH^2 &= GD^2 + HD^2 \\ &= 36 + 89 \\ &= 125 . \\ GH &= 5\sqrt{5} \text{ in.} \end{aligned}$$

$$\begin{aligned} \sin \angle GHD &= \frac{GD}{GH} \\ &= \frac{6}{5\sqrt{5}} \\ &= \frac{6\sqrt{5}}{25} \\ &\simeq .5366 . \end{aligned}$$

Hence,

$$\angle GHD \simeq 32^\circ .$$

EXERCISE 1.1

1. Sketch
   (a) two intersecting planes,
   (b) two skew lines,
   (c) two skew lines and a third line intersecting them,
   (d) a line and a plane intersecting,
   (e) three planes intersecting in a point,
   (f) three planes intersecting in a line.

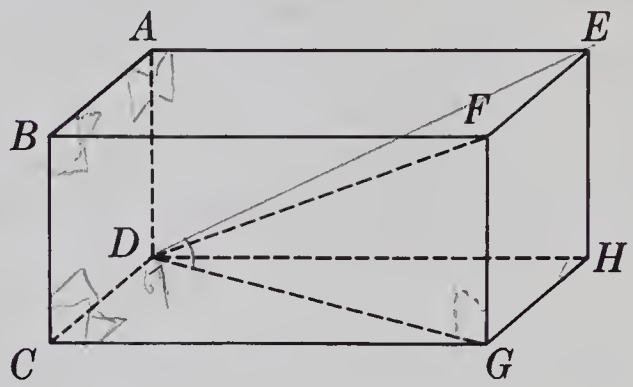
2. How many planes can be passed through (a) a line? (b) two intersecting lines? (c) two skew lines?

3. How many planes can be passed through (a) a point? (b) two points? (c) three points? State any special cases.

4. What is the measure of the dihedral angle between a wall and the floor of your classroom?

5. Calculate the length of the diagonal of a rectangular prism if the lengths of the sides are 12 in., 9 in., and 8 in.

6. In the diagram, which represents a rectangular prism, $CG = 12$ in., $GH = 5$ in., $BC = 6$ in. Calculate the length of the diagonal DF and the measures of $\angle CDG$ and $\angle EDG$.



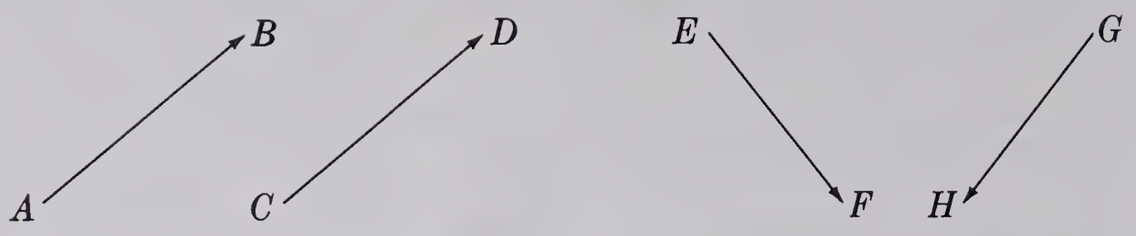
1.2. Vectors

The mass of an object in some particular unit can be expressed by a single real number; the length of a line segment in some particular unit can be expressed by a single real number. Such quantities are known as *scalar* quantities. A scalar quantity has only magnitude and is specified completely in size by a number. In this text all scalars will be real numbers.

However, many physical quantities cannot be completely specified by magnitude alone. A force acting on a body is not completely specified by stating only its magnitude. We must also know the direction in which that force is acting. Such quantities are known as vector quantities and must be specified by a real number called the magnitude, and by a direction in space.

While it is not possible to adequately define a vector, it is possible to understand intuitively the basic idea. Vectors were originally devised for the purpose of dealing with physical quantities such as force and velocity involving both magnitude and direction. However, vectors are now fundamental to many branches of mathematics and, while the application of vectors to physical problems is still of great importance, the application to other subject areas is now of even greater value.

We may describe a nonzero vector as having two fundamental properties: a positive real number called its magnitude or length, and a direction in space. A vector may then be represented by a directed line segment drawn to a suitable scale. Such vectors are usually known as geometric vectors.



In the diagram, AB, CD, EF, and GH represent vectors with A, C, E, and G, the initial points of the line segments. If drawn to the same scale, vectors AB, CD, and GH are all equal in magnitude. AB and CD are also equal in direction (they are parallel) but GH is opposite in direction to AB and CD. The vectors AB and CD are defined to be equal.

DEFINITION. Two geometric vectors are equal if and only if they have the same magnitude and the same direction.

Various symbols are used to name vectors. The vectors in the diagram may be named $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$, $\overrightarrow{GH}$, the arrow indicating that the direction of $\overrightarrow{AB}$ is from A to B. The vector $\overrightarrow{BA}$ is not equal to $\overrightarrow{AB}$ since, although they have the same magnitude, they are opposite in direction. In the diagram,

$$\overrightarrow{AB} = \overrightarrow{CD}, \quad \overrightarrow{AB} \neq \overrightarrow{GH}.$$

Another way of naming vectors in print is by bold-face single letters, **u**, **v**, **w**, etc. This is not too useful in written work but, if the single letter representation is desirable, it may be written as $\vec{u}$, $\vec{v}$, $\vec{w}$, etc., to indicate that we are discussing a vector and not a scalar.

Frequently we are interested in the magnitude of a vector. The magnitude of vector $\overrightarrow{AB}$ or of vector **v** is indicated by

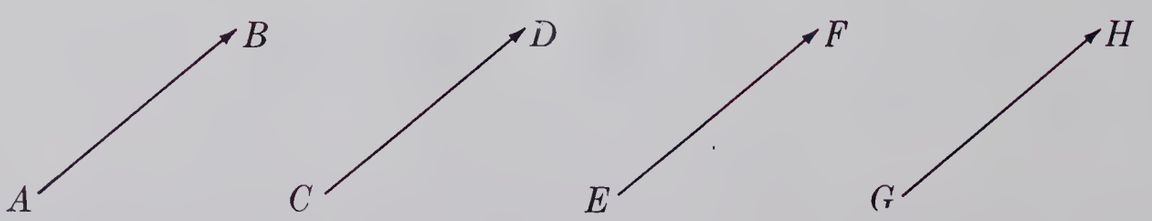
$$|\overrightarrow{AB}| \quad \text{or} \quad |\mathbf{v}|.$$

Note, that in the diagram above, although

$$\overrightarrow{AB} \neq \overrightarrow{GH},$$

it is true that

$$|\overrightarrow{AB}| = |\overrightarrow{GH}|.$$



When we state that $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH}$, we understand from the definition that these vectors have the same magnitude and the same direction. Such vectors

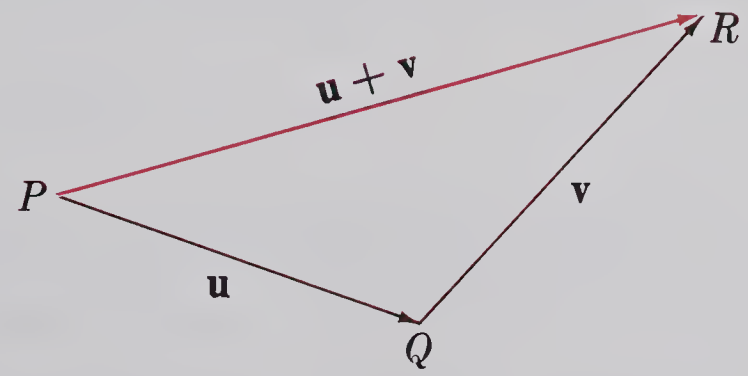
belong to an equivalence class of vectors. By this we understand that in any discussion involving vectors any one of these vectors may be used as a representative of that class.

In previous work we have used any member of the equivalence class of fractions $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \cdots$ as a representative of that class. In any discussion involving fractions, we may substitute any one of $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \cdots$ for any other member of the class; they are equivalent fractions. In the same way, we may substitute any one of $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}, \overrightarrow{GH}, \cdots$ for any other member of the class of equivalent vectors.

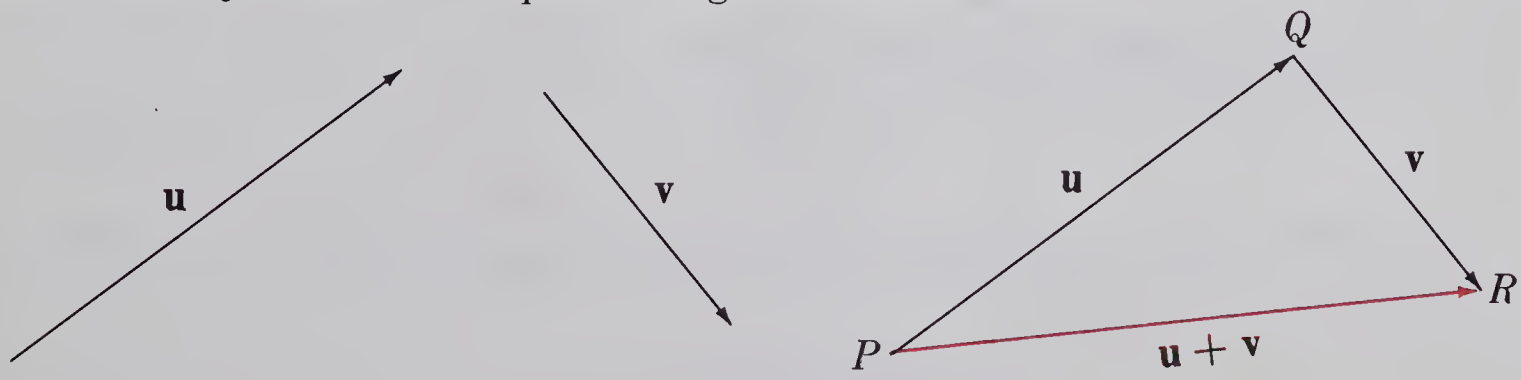
1.3. Addition of Vectors

When the symbols $+, -, \times, \div$ are associated with numbers, we have already assigned a meaning to them. However, when we talk about addition of vectors, the symbols $\overrightarrow{PQ} + \overrightarrow{QR}$, $\mathbf{v} + \mathbf{u}$, $\mathbf{u} - \mathbf{v}$, have, as yet, no meaning, Consequently, we are free to define them as we please but, unless the meaning we assign is of some value, it is unlikely that it will prove acceptable. The actual definition used has arisen because of its application in physics.

DEFINITION. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors represented by $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, respectively, so that the endpoint of the line segment PQ is the initial point of the line segment QR, then the sum of $\mathbf{u}$ and $\mathbf{v}$ is represented by $\overrightarrow{PR}$.



We note that two vectors $\mathbf{u}$ and $\mathbf{v}$ may be represented by any two line segments $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, provided that $\overrightarrow{PQ}$ and $\mathbf{u}$ are equal in magnitude and direction and also that $\overrightarrow{QR}$ and $\mathbf{v}$ are equal in magnitude and direction.

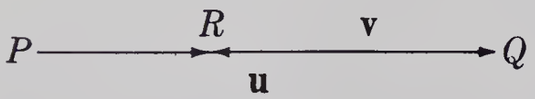


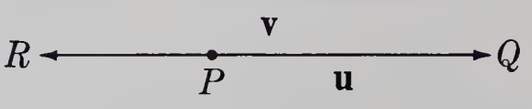
Some special cases are worth noting.

(a) If **u** and **v** have the same direction, then Q lies between P and R and we see an immediate analogy to addition on the real number line.

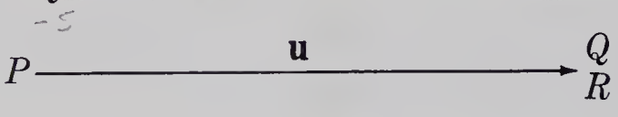
$$\overrightarrow{PR} = \mathbf{u} + \mathbf{v}$$

(b) If **u** and **v** are opposite in direction, then R lies between P and Q or P lies between R and Q and we see the analogy to subtraction on the real number line.





$\overrightarrow{PQ} = \mathbf{u}$ and $\overrightarrow{QR} = \mathbf{v}$ $\qquad$ $\overrightarrow{PR} = \mathbf{u} + \mathbf{v}$



(c) If Q and R coincide so that $\mathbf{v} = \overrightarrow{QR} = \overrightarrow{QQ}$, then

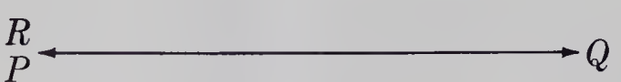
$$\mathbf{u} + \mathbf{v} = \overrightarrow{PQ} + \overrightarrow{QQ} = \overrightarrow{PQ}.$$

Vector **v** or $\overrightarrow{QQ}$ is, in this case, defined to be the zero vector, denoted by **0**, and we can write

$$\mathbf{u} + \mathbf{0} = \mathbf{u}.$$

The analogy with the algebra of real numbers is immediately apparent. Although the zero vector has no direction, it is convenient to regard it as parallel to every vector. The zero vector has magnitude zero.

(d) If $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are equal in magnitude but opposite in direction, P and R will coincide.



Therefore, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} = \mathbf{0}$ or

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}.$$

Since $\overrightarrow{PQ}$ has been denoted by **u**, it would seem appropriate to denote $\overrightarrow{QP}$ by $-\mathbf{u}$ and we can write that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

This equation suggests that we define the negative of a vector as follows.

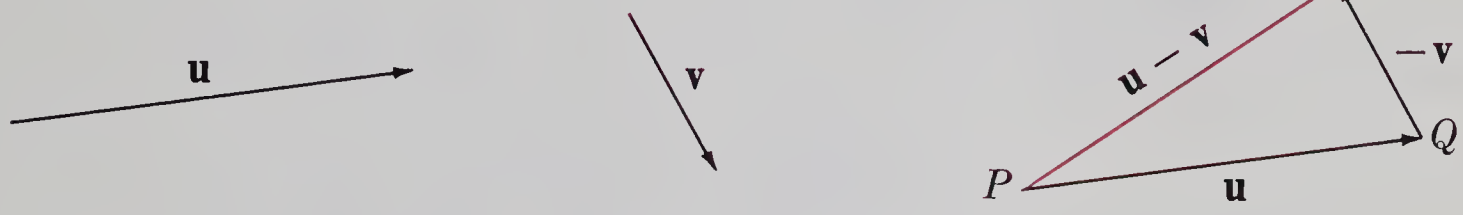
DEFINITION: The negative of a vector $\mathbf{u}$ is a vector equal in magnitude to $\mathbf{u}$ but opposite in direction, and is denoted by $-\mathbf{u}$.

We note that

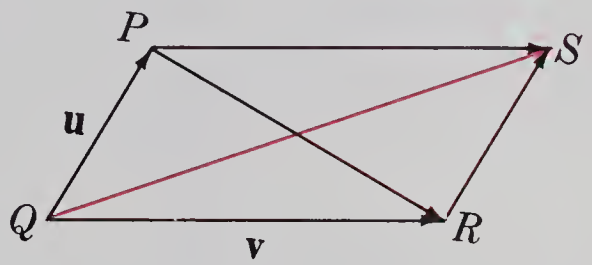
$$-(-\mathbf{u}) = \mathbf{u}\,.$$

This result enables us to define subtraction of vectors in the same way as we define subtraction of real numbers.

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$



If $\overrightarrow{PQ} = \mathbf{u}$ and $\overrightarrow{RQ} = \mathbf{v}$, then $\overrightarrow{QR} = -\mathbf{v}$ and $\overrightarrow{PR} = \mathbf{u} - \mathbf{v}$. Let us consider the parallelogram $PQRS$.



If $\overrightarrow{QP} = \mathbf{u}$ and $\overrightarrow{QR} = \mathbf{v}$, then $\overrightarrow{PS} = \mathbf{v}$ and $\overrightarrow{RS} = \mathbf{u}$. Thus,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \overrightarrow{QP} + \overrightarrow{QR} \\ &= \overrightarrow{QP} + \overrightarrow{PS} \\ &= \overrightarrow{QS}\,.\end{aligned}$$

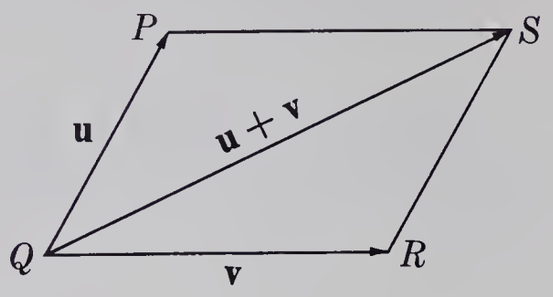
That is, the sum of the vectors can either be obtained from the sides of $\triangle QPS$ or as the diagonal QS of parallelogram $PQRS$. If, instead of drawing the two given vectors so that the endpoint of one is the initial point of the second, as in the triangle method, we prefer to draw both vectors with a common initial point, then we obtain the sum as the diagonal of the parallelogram in which the two given vectors are adjacent sides. The diagonal required will have the same common initial point.

The diagonal PR also represents a vector.

$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PS} + \overrightarrow{SR} \\ &= \mathbf{v} - \mathbf{u}\,. \\ \overrightarrow{RP} &= \overrightarrow{RS} + \overrightarrow{SP} \\ &= \mathbf{u} - \mathbf{v}\,.\end{aligned}$$

Example 1. Two vectors $\mathbf{u}$ and $\mathbf{v}$ have a common initial point and form an angle of 60°. If $|\mathbf{u}| = 6$ and $|\mathbf{v}| = 8$, find $|\mathbf{u} + \mathbf{v}|$ and the angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}$.

Solution: In $\triangle QRS$,



$$QR = 8, \quad RS = 6, \quad \angle QRS = 120^\circ .$$
$$QS^2 = RQ^2 + RS^2 - 2RQ \cdot RS \cos \angle QRS$$
$$= 64 + 36 + 2(8)(6) \cos 60^\circ$$
$$= 148 .$$
$$QS = 2\sqrt{37}$$
$$\simeq 12.2 .$$

Hence, $|\mathbf{u} + \mathbf{v}| \simeq 12.2$.

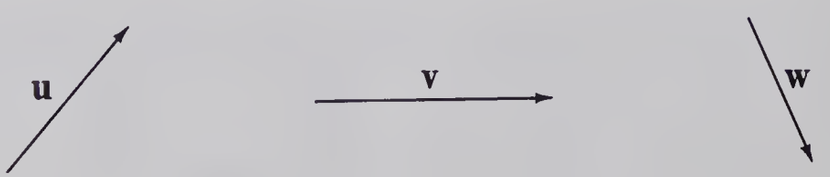
$$\frac{\sin \angle SQR}{RS} = \frac{\sin \angle QRS}{QS} .$$
$$\frac{\sin \angle SQR}{6} = \frac{\sin 120^\circ}{12.2} .$$
$$\sin \angle SQR = \frac{6 \sin 60^\circ}{12.2}$$
$$\simeq \frac{6\sqrt{3}}{12.2 \times 2}$$
$$\simeq .426 .$$

Therefore,

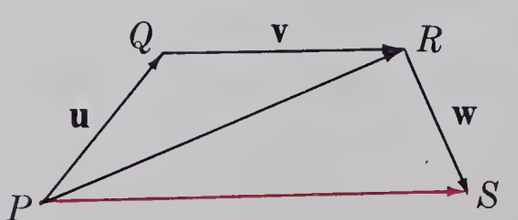
$$\angle SQR \simeq 25^\circ .$$

The angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{v}$ is 25°.

When three or more vectors are involved in a problem they are, in general, not in the same plane. Any two vectors, however, are coplanar, and in certain circumstances, three or more vectors may be coplanar.



If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three coplanar vectors, as in the diagram, they may be represented by the equivalent parallel vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$. Then



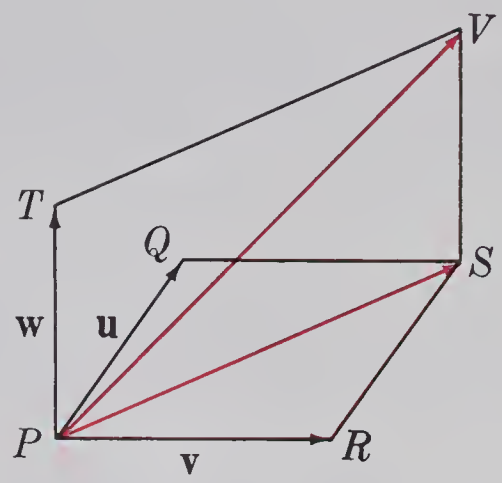
$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$$

and

$$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$$
$$= (\overrightarrow{PQ} + \overrightarrow{QR}) + \overrightarrow{RS}$$
$$= (\mathbf{u} + \mathbf{v}) + \mathbf{w} .$$

In the same way, if **u**, **v**, and **w** are not coplanar, as in the diagram, then

$$\overrightarrow{PV} = (\mathbf{u} + \mathbf{v}) + \mathbf{w}\,.$$



$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are coplanar in the plane $PQSR$ and

$$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{PR}\,.$$

$\overrightarrow{PS}$ and $\overrightarrow{PT}$ are coplanar in the plane $PSVT$ and

$$\overrightarrow{PV} = \overrightarrow{PS} + \overrightarrow{PT}\,.$$

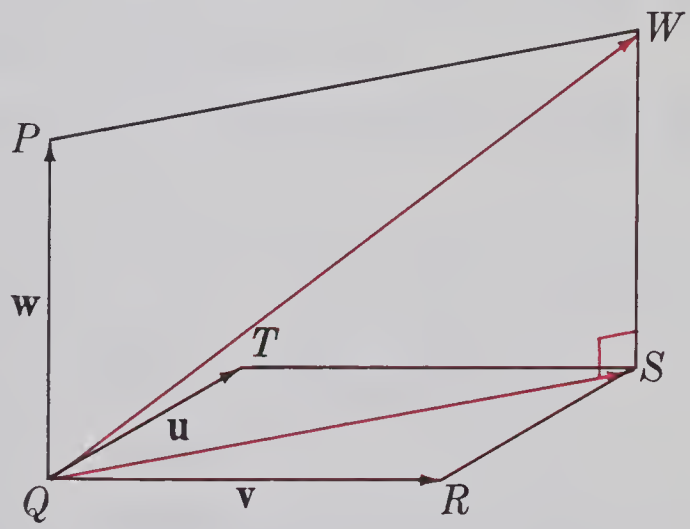
Therefore,

$$\begin{aligned}\overrightarrow{PV} &= \overrightarrow{PS} + \overrightarrow{PT} \\ &= (\overrightarrow{PQ} + \overrightarrow{PR}) + \overrightarrow{PT} \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}\,.\end{aligned}$$

Example 2. Two vectors **u** and **v** have a common initial point and form an angle of 75°. A third vector **w** has the same initial point and is perpendicular to the plane containing **u** and **v**. If $|\,\mathbf{u}\,| = 4$, $|\,\mathbf{v}\,| = 7$, and $|\,\mathbf{w}\,| = 6$, calculate $|\,(\mathbf{u} + \mathbf{v}) + \mathbf{w}\,|$.

Solution:

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (\overrightarrow{QT} + \overrightarrow{QR}) + \overrightarrow{QP} \\ &= \overrightarrow{QS} + \overrightarrow{QP} \\ &= \overrightarrow{QW}\,.\end{aligned}$$



In $\triangle SQR$,

$$\begin{aligned}SQ^2 &= RQ^2 + RS^2 - 2RQ \cdot RS \cos \angle QRS \\ &= 49 + 16 + 2\,(7)\,(4) \cos 75^\circ \\ &\simeq 65 + 56\,(.2588) \\ &\simeq 79.45\,.\end{aligned}$$

In $\triangle WQS$,

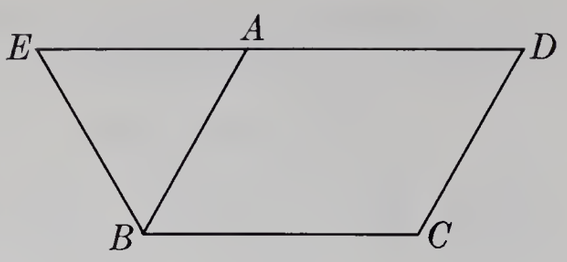
$$\begin{aligned}QW^2 &= QS^2 + SW^2 \\ &\simeq 79.45 + 36 \\ &= 115.45 \\ &\simeq 2\sqrt{29} \\ &\simeq 10.7\,.\end{aligned}$$

Therefore,

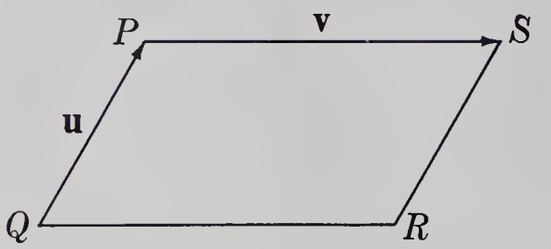
$$|\,(\mathbf{u} + \mathbf{v}) + \mathbf{w}\,| \simeq 11\,.$$

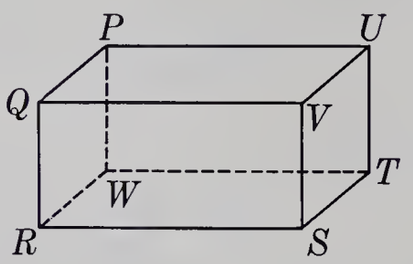
EXERCISE 1.3

1. $ABCD$ is a parallelogram and $EB = AB$. State which vectors (if any) are equal to $\overrightarrow{AB}$, $\overrightarrow{DA}$, $\overrightarrow{EB}$, $\overrightarrow{BC}$, $\overrightarrow{ED}$.



2. $PQRS$ is a parallelogram. $\overrightarrow{QP} = \mathbf{u}$ and $\overrightarrow{PS} = \mathbf{v}$. State which vectors are equal to $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + \mathbf{u}$, $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{u}$.





3. The diagram on the right represents a rectangular prism. State a single vector equal to each of the following.

(a) $\overrightarrow{RQ} + \overrightarrow{RS}$ (b) $\overrightarrow{RQ} + \overrightarrow{QV}$ (c) $\overrightarrow{RW} + \overrightarrow{RS}$
(d) $\overrightarrow{RQ} + \overrightarrow{RT}$ (e) $\overrightarrow{PW} + \overrightarrow{WS}$ (f) $(\overrightarrow{RQ} + \overrightarrow{RS}) + \overrightarrow{VU}$
(g) $\overrightarrow{RQ} + (\overrightarrow{RS} + \overrightarrow{VU})$ (h) $\overrightarrow{PW} - \overrightarrow{VP}$

4. Three vectors $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ form a triangle ABC with $\overrightarrow{AB} = \mathbf{f}$, $\overrightarrow{BC} = \mathbf{g}$, $\overrightarrow{CA} = \mathbf{h}$. State a single vector equal to each of $(\mathbf{f} + \mathbf{g}) + \mathbf{h}$ and $\mathbf{f} + (\mathbf{g} + \mathbf{h})$.

5. If two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, show that

$$|\mathbf{u} + \mathbf{v}| = |\mathbf{u} - \mathbf{v}|.$$

6. Show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$,

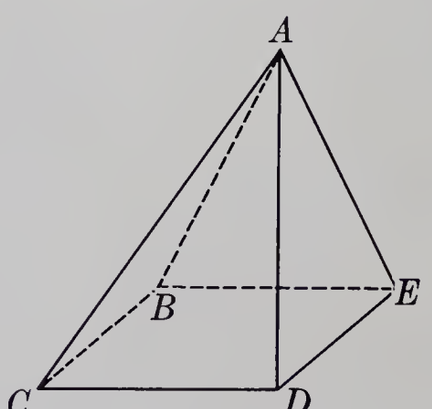
$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

7. If $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CD}$, show that

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

8. The diagram shows a square-based right pyramid. State any vectors equal to each of the following.

(a) $\overrightarrow{CB}$
(b) $\overrightarrow{AB}$
(c) $\overrightarrow{BE}$
(d) $\overrightarrow{AB} + \overrightarrow{BC}$
(e) $\overrightarrow{AB} - \overrightarrow{AE}$
(f) $\overrightarrow{DE} - \overrightarrow{DA}$
(g) $(\overrightarrow{AE} + \overrightarrow{ED}) + \overrightarrow{DC}$
(h) $\overrightarrow{AE} + (\overrightarrow{ED} + \overrightarrow{DC})$



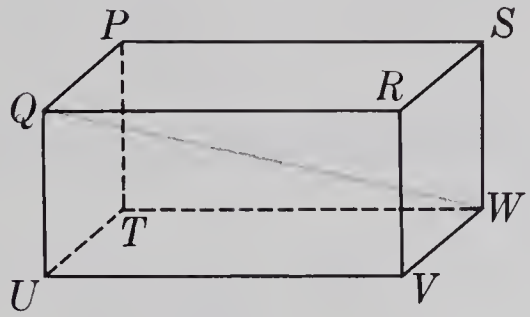
9. Vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are inclined at an angle of 45° and $|\overrightarrow{PQ}| = 6$, $|\overrightarrow{PR}| = 10$. Calculate, to the nearest integer, the value of $|\overrightarrow{PQ} + \overrightarrow{PR}|$ and, to the nearest degree, the angle between $\overrightarrow{PQ} + \overrightarrow{PR}$ and $\overrightarrow{PR}$.

10. Two vectors $\mathbf{u}$ and $\mathbf{v}$ have a common initial point and form an angle of 120°. Calculate, to the nearest integer, the value of $|\mathbf{u} + \mathbf{v}|$ and, to the nearest degree, the angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{v}$, given that $|\mathbf{u}| = 10$ and $|\mathbf{v}| = 12$.

11. The diagram below shows a rectangular prism. If $|\overrightarrow{UQ}| = 3$, $|\overrightarrow{UT}| = 4$, and $|\overrightarrow{UV}| = 12$, calculate

$$|(\overrightarrow{UQ} + \overrightarrow{UT}) + \overrightarrow{UV}|$$

and $\angle SUW$, to the nearest degree.



12. Two vectors $\mathbf{u}$ and $\mathbf{v}$ have a common initial point and form an angle of 60°. A third vector $\mathbf{w}$ has the same initial point and is perpendicular to the plane containing $\mathbf{u}$ and $\mathbf{v}$. If $|\mathbf{u}| = 5$, $|\mathbf{v}| = 6$, and $|\mathbf{w}| = 3$, calculate the value of $|(\mathbf{u} + \mathbf{v}) + \mathbf{w}|$, to the nearest integer.

1.4. Algebraic Properties of Vector Addition

Algebraic addition of real numbers has certain properties with which we are familiar. If a, b, and c are real numbers, then

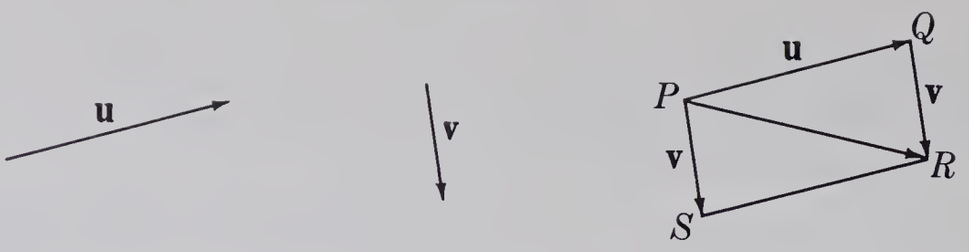
$$\begin{aligned} a + b &= b + a && \text{Commutative property,} \\ (a + b) + c &= a + (b + c) && \text{Associative property,} \\ a + 0 &= 0 + a = a && \text{Identity property,} \\ a + (-a) &= (-a) + a = 0 && \text{Inverse property.} \end{aligned}$$

From the work of the previous sections it would appear that vectors also have this set of properties.

1. *Commutative Property* If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

As with the commutative property of addition of numbers, this is really an axiom of vector addition. However, its plausibility can be illustrated intuitively by letting $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{QR}$ and using our definition of vector addition.



$$\mathbf{u} + \mathbf{v} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}\,.$$
$$\mathbf{v} + \mathbf{u} = \overrightarrow{PS} + \overrightarrow{SR} = \overrightarrow{PR}\,.$$

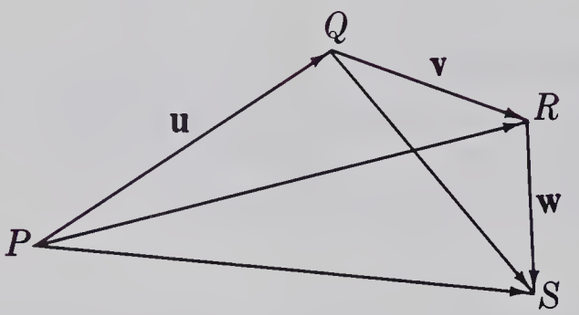
Therefore,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}\,.$$

2. *Associative Property*

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})\,.$$

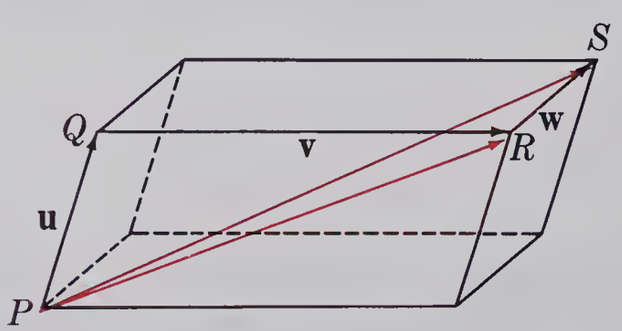
Again this is an axiom for vector addition but it may be illustrated by the addition of geometrical vectors in a diagram.



$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (\overrightarrow{PQ} + \overrightarrow{QR}) + \overrightarrow{RS} \\ &= \overrightarrow{PR} + \overrightarrow{RS} \\ &= \overrightarrow{PS}\,.\end{aligned}$$
$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= \overrightarrow{PQ} + (\overrightarrow{QR} + \overrightarrow{RS}) \\ &= \overrightarrow{PQ} + \overrightarrow{QS} \\ &= \overrightarrow{PS}\,.\end{aligned}$$

Therefore,

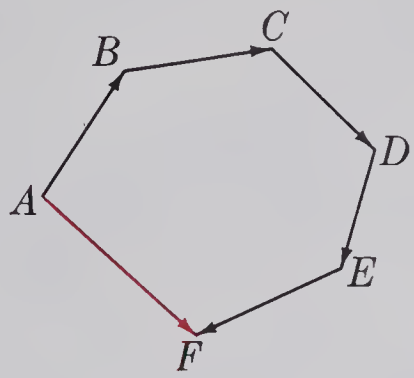
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})\,.$$



Note that this last equation applies whether the vectors are coplanar or not. Both diagrams indicate the same result. Remember that, while any two vectors are coplanar, three vectors, in general, are not coplanar.

These two properties indicate that vector addition may be performed in any order. The brackets in $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ are no longer necessary and may now be omitted.

It should also be noted that this may be extended to include any number of vectors in any number of planes.



In the diagram,

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{AF}.$$

Remember again that although the diagram is drawn on a sheet of paper, the vectors need not be coplanar. It is useful to make a three dimensional "diagram" or model of wire, or drinking straws, or pipe cleaners.

3. *Identity Property* In Section 1.3, we have defined the zero vector $\mathbf{0}$ and, by our definition, this is the identity element for vector addition.

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}.$$

4. *Inverse Property* Again in Section 1.3, we have defined the negative of a vector. If $\mathbf{u} = \overrightarrow{PQ}$, then $-\mathbf{u} = \overrightarrow{QP}$, and

$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}.$$

Example 1. Prove that

$$(\mathbf{u} + \mathbf{v}) + (\mathbf{w} + (-\mathbf{u})) = \mathbf{v} + \mathbf{w}.$$

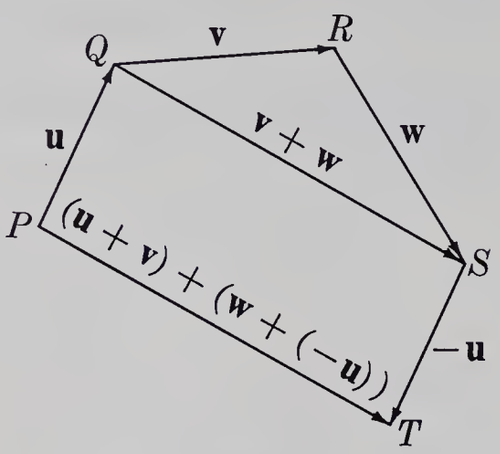
Solution:

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + (\mathbf{w} + (-\mathbf{u})) &= (\mathbf{v} + \mathbf{u}) + (-\mathbf{u} + \mathbf{w}) && (1)\\
&= [(\mathbf{v} + \mathbf{u}) + (-\mathbf{u})] + \mathbf{w} && (2)\\
&= [\mathbf{v} + (\mathbf{u} + (-\mathbf{u}))] + \mathbf{w} && (3)\\
&= [\mathbf{v} + \mathbf{0}] + \mathbf{w} && (4)\\
&= \mathbf{v} + \mathbf{w}. && (5)
\end{aligned}$$

In step (1) we used the commutative property twice, in steps (2) and (3) we made use of the associative property, in step (4) we used the inverse property, and in step (5), the identity property.

It should be mentioned that, while we proved this identity in detail using only the four properties, it would be more usual to eliminate most of the steps as we do in proving algebraic identities.

The result may be indicated diagrammatically.

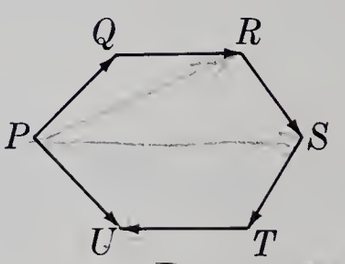


$$\overrightarrow{QS} = \mathbf{v} + \mathbf{w}.$$
$$\overrightarrow{PT} = (\mathbf{u} + \mathbf{v}) + (\mathbf{w} + (-\mathbf{u})).$$

$PQST$ is a parallelogram; therefore

$$\overrightarrow{QS} = \overrightarrow{PT}.$$

EXERCISE 1.4



1. Using the diagram, show that
$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} + \overrightarrow{TU} = \overrightarrow{PU}$.

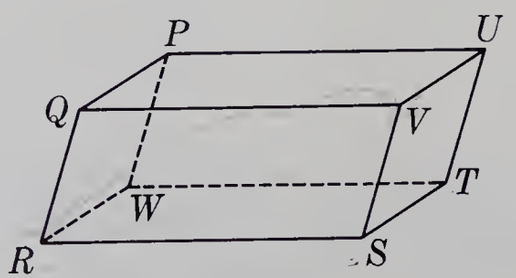
2. Using the diagram of question (1), show that
$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} + \overrightarrow{TU} + \overrightarrow{UP} = \mathbf{0}$.

3. Prove that
$$(\mathbf{u} + \mathbf{v}) + (-\mathbf{u}) = \mathbf{v}.$$

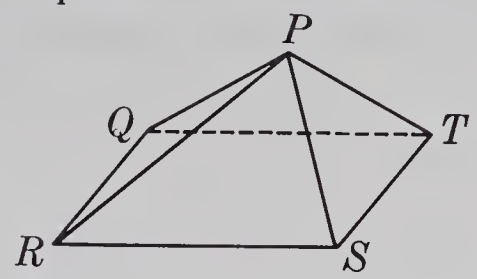
4. Prove that for given vectors $\mathbf{u}$ and $\mathbf{v}$, there is always a vector $\mathbf{w}$ such that
$$\mathbf{u} + \mathbf{w} = \mathbf{v}.$$

5. Prove that
$$\mathbf{u} + \mathbf{v} + (-\mathbf{u}) + \mathbf{w} + (-\mathbf{v}) = \mathbf{w}.$$

6. From the diagram, state a single vector equal to each of the following.
(a) $\overrightarrow{RQ} + \overrightarrow{QV} + \overrightarrow{VS} + \overrightarrow{ST}$
(b) $\overrightarrow{RS} + \overrightarrow{RQ} + \overrightarrow{VP} + \overrightarrow{PW}$
(Note that all opposite planes are parallel.)



7. From the diagram, state a single vector equal to each of the following.
   (a) $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS}$
   (b) $\overrightarrow{PS} + \overrightarrow{ST} + \overrightarrow{TQ} + \overrightarrow{QR} + \overrightarrow{RP}$
   (c) $\overrightarrow{PR} - \overrightarrow{PQ}$
   (d) $\overrightarrow{PQ} - \overrightarrow{PR}$

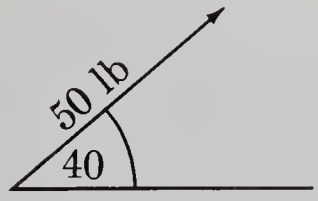


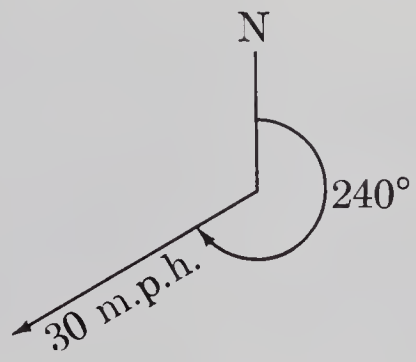
1.5. Force and Velocity

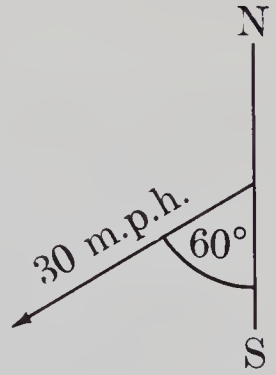
The mathematics of vectors was first developed to simplify working with such physical quantities as force and velocity where both magnitude and direction are of importance. We will now look at some examples of this application.

To represent a physical vector, we draw an arrow whose length and direction indicate the magnitude and direction of the vector.

A force of 50 pounds acting on a body at an angle of 40° to the horizontal would be represented by a vector drawn at an angle of 40° to the horizontal. If we decide that a suitable scale is 1 inch to represent 10 pounds, then the length of the vector would be 5 inches. A velocity of 30 m.p.h. at a bearing of 240° would be represented by a vector of magnitude 30 on some suitable scale. The 240° angle is, by convention, measured in a clockwise direction from north. This latter vector may also be given as a velocity of 30 m.p.h. at an angle of S 60° W indicating that the velocity is in a direction of 60° to the west of south.

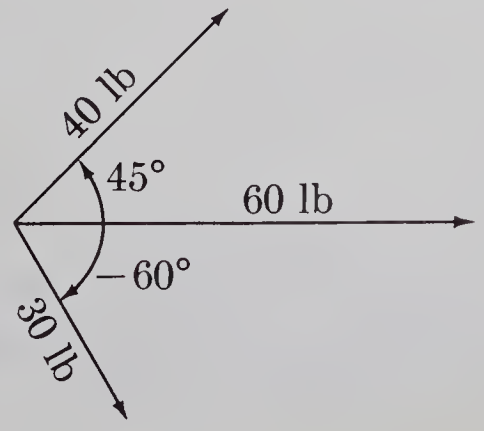






When two or more forces act on a body, the resultant effect depends on the magnitude and direction of the given forces.

The diagram indicates three forces acting at a point on a body: a force of 60 pounds acting in a horizontal direction, a force of 40 pounds acting at an angle of 45° to the horizontal, and a force of 30 pounds acting at an angle of −60° to the horizontal.

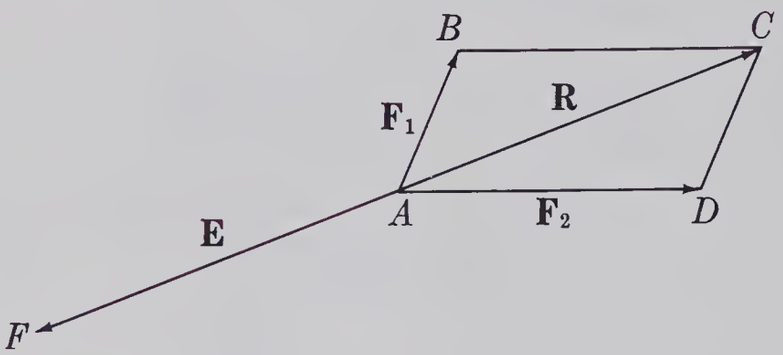


The resultant of two or more forces acting at a point on a body may be defined in terms of the forces or in terms of the associated vectors.

DEFINITION. The resultant of two or more forces acting at a point on a body is that single force which would have the same effect as the given forces. The resultant of two or more physical vectors representing forces with a common initial point is the sum of the given vectors.

DEFINITION. The equilibrant of two or more forces acting at a point on a body is that single force which, when applied to the body, maintains it in a state of equilibrium.

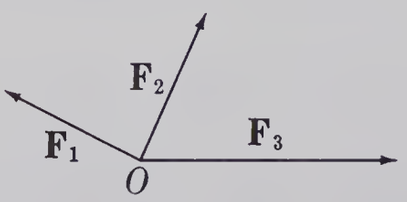
In the diagram, $\mathbf{F}_1$ and $\mathbf{F}_2$ are two forces represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$, acting on a body at A.

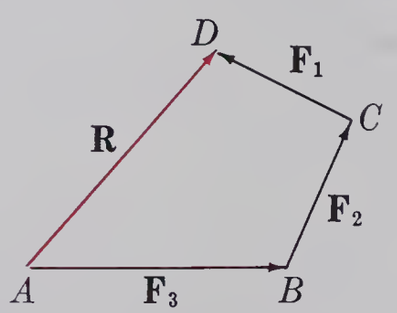


$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}.$$

Hence vector $\overrightarrow{AC}$ represents in magnitude and direction the resultant $\mathbf{R}$ of $\mathbf{F}_1$ and $\mathbf{F}_2$. The body would tend to move in the direction represented by $\overrightarrow{AC}$. If a force equal in magnitude to $\mathbf{R}$ were applied at A, but in the opposite direction, it would counteract the resultant $\mathbf{R}$ and the body would be maintained in a state of equilibrium. Such a force is the equilibrant $\mathbf{E}$ and is represented by $\overrightarrow{AF}$ where $\overrightarrow{AF} = -\overrightarrow{AC}$.

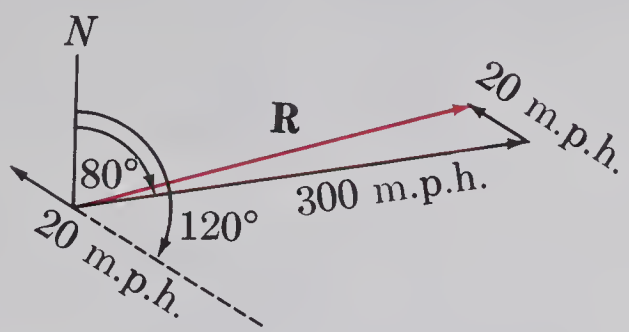
The equilibrant of two or more forces acting at a point is equal in magnitude but opposite in direction to the resultant.



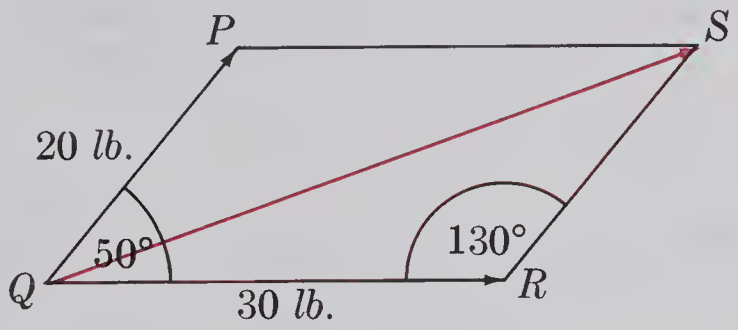


In the diagram, three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ act on a body at O. These forces are represented by $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CD}$. By definition, the resultant $\mathbf{R}$ is represented by $\overrightarrow{AD}$ and the equilibrant by $\overrightarrow{DA}$ or $-\overrightarrow{AD}$.

If an aircraft is flying on a bearing of 80° at 300 m.p.h., and a wind is blowing from 120° at 20 m.p.h., as shown in the diagram, the resultant **R** of the two velocities will give the actual ground speed and direction of the aircraft.



Example 1. Two forces of 20 pounds and 30 pounds act on a body at an angle of 50° to each other. Find the magnitude and direction of the resultant, to the nearest pound and the nearest degree.



Solution: In $\triangle PQR$,

$$\begin{aligned} QS^2 &= RQ^2 + RS^2 - 2RQ \cdot RS \cos \angle QRS \\ &= 900 + 400 - 2\,(30)\,(20) \cos 130° \\ &= 1300 + 1200 \cos 50° \\ &\simeq 1300 + 1200\,(.6428) \\ &= 2071.36\,. \end{aligned}$$

Therefore,

$$QS \simeq 46\,.$$

$$\frac{\sin \angle SQR}{RS} = \frac{\sin \angle SRQ}{QS}\,.$$

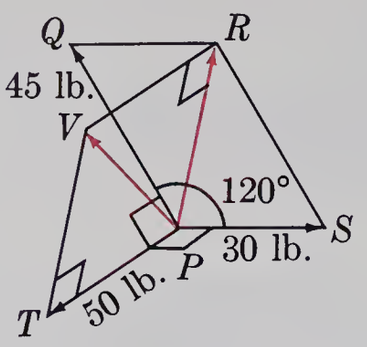
$$\begin{aligned} \sin \angle SQR &\simeq \frac{20 \sin 50°}{46} \\ &\simeq \frac{20\ (.7660)}{46} \\ &\simeq .333\,. \end{aligned}$$

Thus,

$$\angle SQR \simeq 19°\,.$$

Therefore, the resultant force has a magnitude of 46 pounds and acts at an angle of 19° to the 30-pound force.

Example 2. Three forces of 30 pounds, 45 pounds, and 50 pounds act on a body. The first two act at an angle of 120° to each other and the third is perpendicular to the plane of the first two. Find, to the nearest pound, the magnitude of the resultant and the angle which the resultant makes with the plane containing the first two forces.



Solution: In $\triangle PRS$,

$$\begin{aligned} PR^2 &= SP^2 + SR^2 - 2SP \cdot SR \cos \angle RSP \\ &= 900 + 2025 - 2(30)(45) \cos 60° \\ &= 2925 - 1350 \\ &= 1575 . \end{aligned}$$

In $\triangle PTV$, $\angle PTV = 90°$.

$$\begin{aligned} PV^2 &= PT^2 + TV^2 \\ &= PT^2 + PR^2 \\ &= 2500 + 1575 \\ &= 4075 . \end{aligned}$$

Therefore,

$$PV \simeq 64 .$$

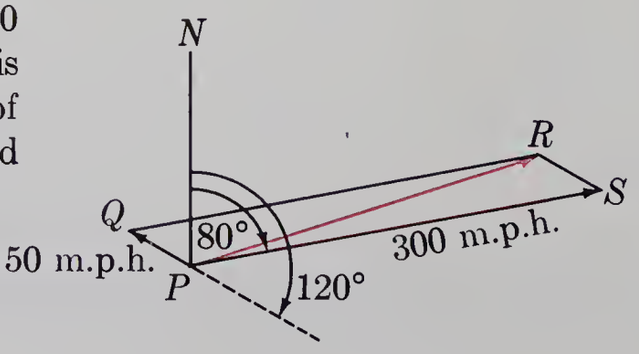
$$\begin{aligned} \sin \angle VPR &= \frac{VR}{PV} \\ &= \frac{50}{64} \\ &\simeq .781 . \end{aligned}$$

Therefore,

$$\angle VPR \simeq 51° .$$

Therefore, the resultant is a force of 64 pounds acting at an angle of 51° to the plane containing the first two forces.

Example 3. An aircraft is flying at 300 m.p.h. on a bearing of 80°. A wind is blowing at 50 m.p.h. from a bearing of 120°. Determine the actual ground speed and direction of the aircraft.



Solution: In $\triangle PRS$, $\angle RSP = 40°$.

$$\begin{aligned} PR^2 &= SP^2 + SR^2 - 2SP \cdot SR \cos \angle RSP \\ &= 90000 + 2500 - 2\,(300)\,(50) \cos 40° \\ &= 92500 - 30000\,(.7660) \\ &= 92500 - 22{,}980 \\ &= 69{,}520\,. \end{aligned}$$

Therefore,

$$PR \simeq 264\,.$$

$$\frac{\sin \angle RPS}{RS} = \frac{\sin \angle RSP}{PR}\,.$$

$$\begin{aligned} \sin \angle RPS &= \frac{50 \sin 40°}{264} \\ &\simeq \frac{50\ (.6428)}{264} \\ &\simeq .1217\,. \end{aligned}$$

Thus,

$$\angle RPS \simeq 7°\,.$$

Therefore, the ground speed is 264 m.p.h. on a bearing of 73°.

EXERCISE 1.5

1. Calculate the magnitude of the resultant force of each of the following pairs of forces. All forces should be given to the nearest pound and all angles to the nearest degree.
   (a) 30 pounds and 40 pounds at an angle of 90° to each other
   (b) 40 pounds and 40 pounds at an angle of 60° to each other
   (c) 50 pounds and 80 pounds at an angle of 120° to each other
   (d) 10 pounds and 12 pounds at an angle of 30° to each other
   (e) 20 pounds and 15 pounds at an angle of 180° to each other
   (f) 120 pounds and 100 pounds at an angle of 100° to each other
2. Two forces act on a body: a force of 50 pounds at an angle of 30° to the horizontal, and a force of 20 pounds at an angle of −90° to the horizontal. Calculate the magnitude of the resultant force and the angle which it makes with the horizontal.
3. Two forces act on a body: a force of 12 pounds at an angle of 60° to the horizontal, and a force of 20 pounds at an angle of 120° to the horizontal. Calculate the magnitude and direction of the equilibrant force.
4. Two equal forces acting at an angle of 90° to each other have a resultant force of 40 pounds. Calculate the magnitude of the two forces.

5. Two equal forces acting at an angle of 150° to each other have a resultant force of 50 pounds. Calculate the magnitude of the two forces.

6. Three coplanar forces act on a body: a force of 30 pounds at an angle of 30° to the horizontal, a force of 40 pounds at an angle of 120° to the horizontal, and a force of 20 pounds at an angle of −90° to the horizontal. Calculate the magnitude of the resultant force and the angle which it makes with the horizontal.

7. Three coplanar forces of x pounds, 10 pounds, and 20 pounds act on a body and maintain it in a state of equilibrium. If the x-pound force acts along the horizontal and the 10-pound force at an angle of 90°, find the value of x and the direction of the 20-pound force.

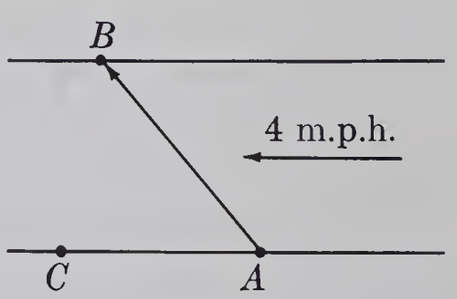
8. Three coplanar forces of x pounds, 10 pounds, and 20 pounds act on a body and maintain it in a state of equilibrium. If the x-pound force acts along the horizontal and the 10-pound force at an angle of 60°, find the value of x and the direction of the 20-pound force.

9. An aircraft is flying at 500 m.p.h. on a bearing of 60°. A 60 m.p.h. wind is blowing from a bearing of 90°. Calculate the ground speed and direction of the aircraft.

10. An aircraft is flying at 600 m.p.h. on a bearing of 60°. An 80 m.p.h. wind is blowing from a bearing of 270°. Calculate the ground speed and direction of the aircraft.

11. A ship is sailing on a compass bearing of 320° at a speed of 20 knots. A current of 6 knots is travelling from a compass bearing of 230°. Calculate the actual direction of travel and the actual speed of the ship.

12. A fisherman wishes to cross a river from A to B as shown in the diagram. $\angle BAC = 50°$ and his outboard motor produces a speed of 10 m.p.h. If there is a current of 4 m.p.h. downstream as shown, in what direction should he travel and what will be his actual speed?

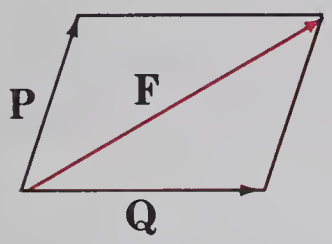


13. An aircraft pilot wishes to fly from A to B. The distance is 400 miles on a bearing of 160° and the cruising speed of the aircraft is 350 m.p.h. If there is a wind of 50 m.p.h. blowing from a bearing of 280°, what heading should the pilot take in order to reach his destination and how long will the flight take, to nearest minute?

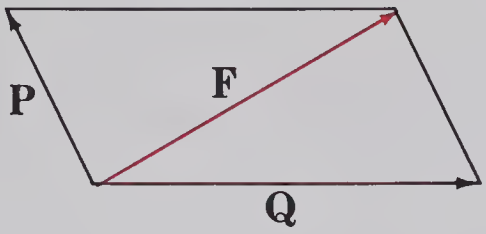
14. Three forces act on a body: a force of 10 pounds and a force of 20 pounds at an angle of 60° to each other, and a force of 30 pounds perpendicular to the plane containing the first two forces. Calculate the magnitude of the resultant force and the angle which it makes with the plane containing the first two forces.

15. Three forces act on a body: a force of 20 pounds and a force of 30 pounds at an angle of 120° to each other, and a force of 30 pounds perpendicular to the plane of the first two forces. Calculate the magnitude of the resultant force and the angle which it makes with the plane of the first two forces.

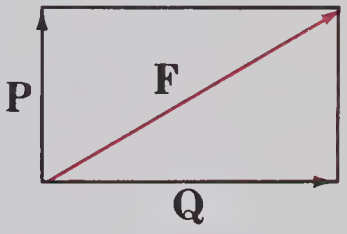
1.6. Resolution of Forces



(*i*)



(*ii*)

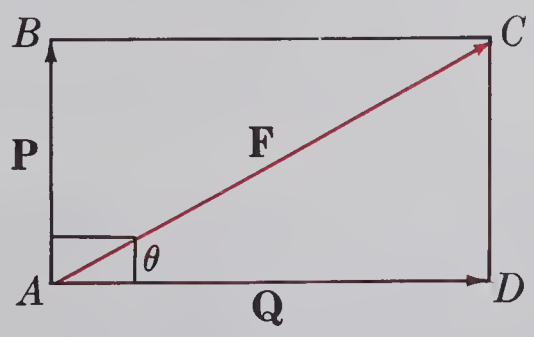


(*iii*)

In the diagram, two forces **P** and **Q** have a force **F** as their resultant. All three resultants are equal in magnitude and direction. Two given forces have only one resultant but one given force may be the resultant of any number of different pairs of given forces. **F** is the resultant of **P** and **Q**; **P** and **Q** are the components of **F**. A given force has an unlimited number of possible components in a given plane. In Figure (*iii*), the components are perpendicular to each other. Such components are called rectangular components. Again it is possible to have an unlimited number of pairs of rectangular components of a given force. However, if the angle between the given force and one of its rectangular components is given, then only one pair of rectangular components is possible.

In studying forces it is very often of value to find two rectangular components of the given force. The force is then said to have been resolved into rectangular components. While it is possible to resolve a force into nonrectangular components, this is rarely required and, unless stated otherwise, we shall assume that when we state that a force has been resolved into components, we mean rectangular components. As well it should be noted that it is also possible to resolve a force into more than two components not necessarily in the same plane.

In the diagram below, a force **F** has been resolved into rectangular components **P** and **Q** so that the angle between **F** and **Q** is θ.



Thus,

$$\frac{AD}{AC} = \cos\theta\,.$$

$$AD = AC \cos\theta .$$

Hence,

$$|\mathbf{Q}| = |\mathbf{F}| \cos\theta .$$

$$\frac{AB}{AC} = \cos\,(90° - \theta) .$$

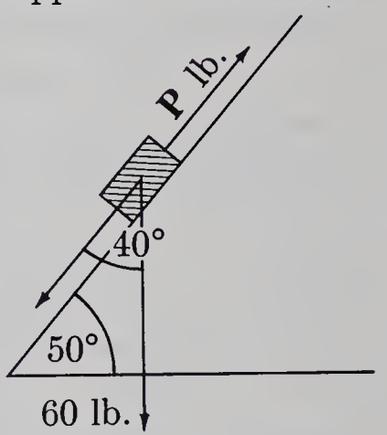
$$AB = AC \sin\theta .$$

Therefore,

$$|\mathbf{P}| = |\mathbf{F}| \sin\theta .$$

We see that the component of a force **F**, resolved at an angle of θ to the force, has magnitude $|\mathbf{F}| \cos\theta$, and that the magnitude of the associated rectangular component is $|\mathbf{F}| \sin\theta$.

Example 1. A body weighing 60 pounds rests on a smooth ramp which is inclined at an angle of 50° to the horizontal. What force must be applied in the direction of the ramp in order to maintain the body at rest?

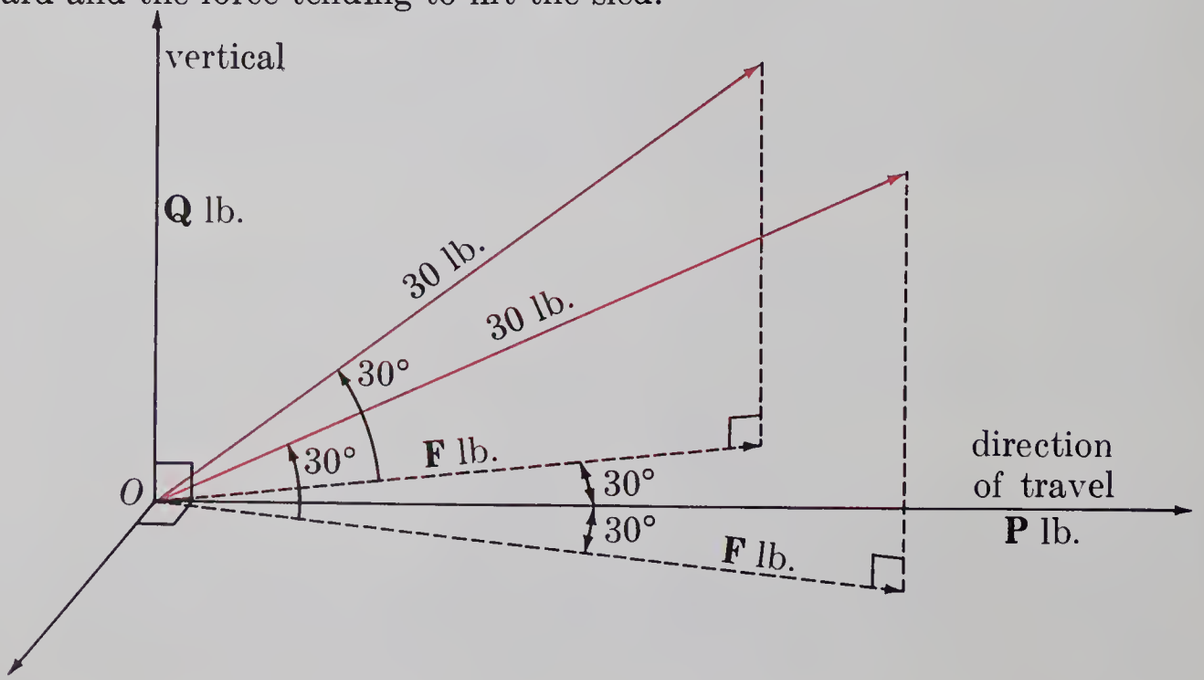


Solution: Let the required force be **P** pounds. To maintain equilibrium,

$$\begin{aligned} |\mathbf{P}| &= 60 \cos 40° \\ &\simeq 60\,(.7660) \\ &= 45.96 . \end{aligned}$$

Therefore, the required force is 46 pounds in the direction of the ramp.

Example 2. Two boys are pulling a sled along smooth level ice. Each boy is exerting a pull of 30 pounds at an angle of 30° to the horizontal and at an angle of 30° on either side of the direction of travel. Find the force that draws the sled forward and the force tending to lift the sled.



Solution: Let the force drawing the sled forward be **P** pounds. Let the force tending to lift the sled be **Q** pounds. Let the horizontal component of each applied force be **F** pounds.

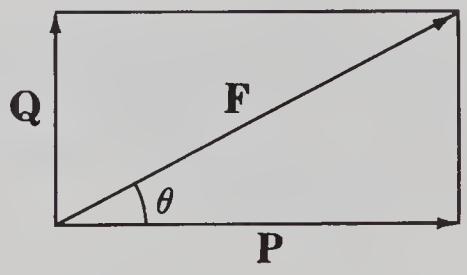
$$
\begin{aligned}
|\mathbf{F}| &= 30 \cos 30^\circ . \\
|\mathbf{P}| &= |\mathbf{F}| \cos 30^\circ + |\mathbf{F}| \cos 30^\circ \\
&= 2\,|\mathbf{F}| \cos 30^\circ \\
&= 2(30 \cos 30^\circ)\ (\cos 30^\circ) \\
&= 60 \cos^2 30^\circ \\
&= \frac{60 \times 3}{4} \\
&= 45 . \\
|\mathbf{Q}| &= 30 \sin 30^\circ + 30 \sin 30^\circ \\
&= 15 + 15 \\
&= 30 .
\end{aligned}
$$

Therefore, the force drawing the sled forward is 45 pounds and the force tending to lift the sled is 30 pounds.

EXERCISE 1.6

Give all solutions correct to the nearest unit.

1. In the diagram, **P** and **Q** are rectangular components of the force **F**. Find **P** and **Q** for each of the following values for **F** and θ.

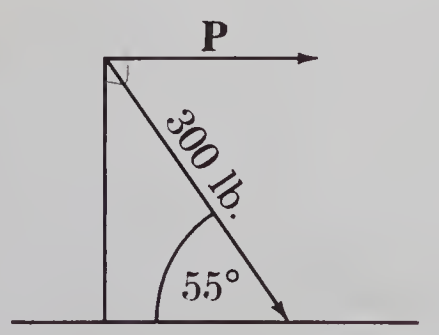


(a) $|\mathbf{F}| = 40$ pounds, $\theta = 30^\circ$
(b) $|\mathbf{F}| = 60$ pounds, $\theta = 45^\circ$
(c) $|\mathbf{F}| = 200$ pounds, $\theta = 60^\circ$
(d) $|\mathbf{F}| = 20$ pounds, $\theta = 35^\circ$
(e) $|\mathbf{F}| = 72$ pounds, $\theta = 54^\circ$
(f) $|\mathbf{F}| = 45$ pounds, $\theta = 20^\circ$

2. Resolve a 35-pound force into two rectangular components so that the magnitudes of the components are in the ratio 3 : 4. Calculate the angle between the 35-pound force and the larger component.

3. A boy pulls a sled along smooth level ice exerting a force of 20 pounds at an angle of 45° to the horizontal. What is the magnitude of the force which moves the sled forward and what is the magnitude of the force tending to lift the sled?

4. A telephone pole is anchored by means of a guy wire as shown in the sketch. What is the magnitude of the force **P**, perpendicular to the pole, if the tension in the guy wire is 300 pounds and the wire makes an angle of 55° with the ground?



5. A 100-pound trunk rests on a smooth ramp inclined at an angle of 25° to the horizontal. Find the force which must be applied parallel to the ramp to keep the trunk stationary on the ramp.

6. Two horses are hauling a load of logs along a smooth level road. If each horse is exerting a pull of 500 pounds at an angle of 25° to the horizontal and at an angle of 30° on each side of the direction of travel, find the force moving the load along the road.

7. A force of 12 pounds is resolved into two components of 8 pounds and 7 pounds (not necessarily rectangular). Find the angle which each component makes with the 12-pound force.

8. Resolve a force of 50 pounds into two equal rectangular components.

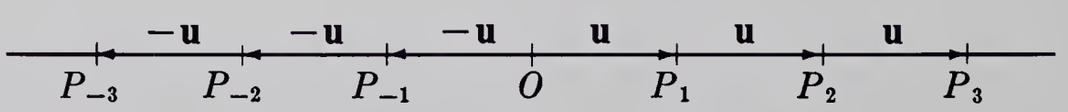
9. A smooth road is inclined at an angle of 20° to the horizontal. What force must be applied at an angle of 20° to the road, to maintain a sled weighing 50 pounds in equilibrium on the road?

10. An aircraft is climbing at an angle of 15° to the horizontal and at a speed of 400 m.p.h. What is its rate of climb in feet per minute and its horizontal ground speed in miles per hour?

11. A rocket is moving at an angle of 60° to the horizontal and at a speed of 10,000 m.p.h. What is its rate of climb in feet per minute and its horizontal ground speed in miles per hour?

12. An aircraft headed due north is climbing at an angle of 20° and at an air speed of 400 m.p.h. It is affected by a horizontal easterly wind of 60 m.p.h. What is the rate of climb in feet per minute and the horizontal ground speed in miles per hour?

1.7. Multiplication by a Scalar



Let us consider the vector $\mathbf{u} = \overrightarrow{OP_1}$ in the diagram.

$$\overrightarrow{OP_2} = \mathbf{u} + \mathbf{u},$$
$$\overrightarrow{OP_3} = \mathbf{u} + \mathbf{u} + \mathbf{u},$$

and it would seem appropriate to represent $\overrightarrow{OP_3}$ by $3\mathbf{u}$ as in the algebra of real numbers. Similarly, it would seem natural to represent $\overrightarrow{OP_{-3}}$ by $(-3)\mathbf{u}$. In general,

$$\overrightarrow{OP}_n = \mathbf{u} + \mathbf{u} + \mathbf{u} + \cdots + \mathbf{u} \qquad (n \text{ terms})$$
$$= n\mathbf{u}\,.$$

If n is a positive scalar, then $n\mathbf{u}$ is a vector in the same direction as $\mathbf{u}$ but with magnitude n times as great. If n is a negative scalar, then $n\mathbf{u}$ is a vector in the opposite direction to that of $\mathbf{u}$ and with magnitude $|\,n\,|$ times the magnitude of $\mathbf{u}$.

This suggests the following definition of the multiplication of a vector by a scalar.

DEFINITION. If $\mathbf{u}$ is a *nonzero* vector and k, a *nonzero* real number, the vector $k\mathbf{u}$ is defined by the following rules.

(1) $|\,k\mathbf{u}\,| = |\,k\,|\,|\,\mathbf{u}\,|$.

(2) The directions of $k\mathbf{u}$ and $\mathbf{u}$ are the same if $k > 0$ and opposite if $k < 0$.

This definition may be extended to include the zero vector or the scalar zero.

DEFINITION. $k\mathbf{0} = \mathbf{0}$ and $0\mathbf{u} = \mathbf{0}$.

That is, the product of any real number and the zero vector is the zero vector, and the product of the number zero and any vector is the zero vector.

Properties of Multiplication by a Scalar

For the following properties, $\mathbf{u}$ and $\mathbf{v}$ are vectors and k and l are real numbers.

(1) $k\mathbf{u}$ is a vector.

(2) $k\mathbf{u} = \mathbf{u}k$.

(3) If $k = l$ and $\mathbf{u} = \mathbf{v}$, then $k\mathbf{u} = l\mathbf{v}$.

(4) $(kl)\mathbf{u} = k(l\mathbf{u})$.

(5) $1\mathbf{u} = \mathbf{u}$.

(6) $(-1)\mathbf{u} = -\mathbf{u}$.

(7) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

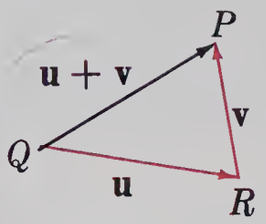
(8) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$.

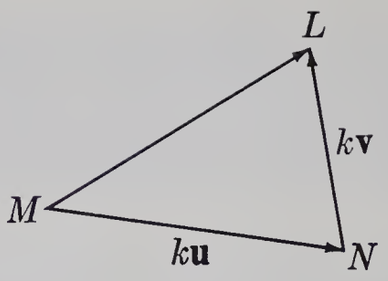
(9) $|\,k\mathbf{u}\,| = |\,k\,|\;|\,\mathbf{u}\,|$.

We note that there are two distributive properties. Property (7) states that multiplication by a scalar is distributive over vector addition; property (8) states that multiplication by a given vector is distributive over scalar addition. The validity of these properties may be demonstrated by using geometric vectors. We shall demonstrate some of them and leave the others as exercises.

Property (7) $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

Case 1. $k > 0$.





Let $\overrightarrow{QR} = \mathbf{u}$ and $\overrightarrow{RP} = \mathbf{v}$; then $\overrightarrow{QP} = \mathbf{u} + \mathbf{v}$.
Construct $\triangle LMN$ with $MN \parallel QR$, $NL \parallel RP$, $ML \parallel QP$, and $\overrightarrow{MN} = k\mathbf{u}$. Then

$$\triangle LMN \,|||\, \triangle PQR .$$

Therefore,

$$\overrightarrow{NL} = k\,\overrightarrow{RP} = k\mathbf{v}$$

and

$$\overrightarrow{ML} = k\,\overrightarrow{QP} = k(\mathbf{u}+\mathbf{v}) .$$

But

$$\overrightarrow{ML} = \overrightarrow{MN} + \overrightarrow{NL}$$

$$= k\mathbf{u} + k\mathbf{v} .$$

Therefore,

$$k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v} .$$

The following cases are also possible and demonstrations of these are asked for in Exercise 1.7.

Case 2. $k < 0$.

Case 3. $k = 0$.

Case 4. $\mathbf{u}$ and $\mathbf{v}$ are collinear.

Property (8) $(k+l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$.

Case 1. $k > 0, \quad l > 0, \quad \mathbf{u} \neq \mathbf{0}$.

Since $k > 0, l > 0$, vectors $k\mathbf{u}$ and $l\mathbf{u}$ have the same direction as $\mathbf{u}$. Also,

$$|k\mathbf{u}| + |l\mathbf{u}| = |k|\,|\mathbf{u}| + |l|\,|\mathbf{u}|$$
$$= |k+l|\,|\mathbf{u}|. \qquad (|k|, |l|, |\mathbf{u}| \text{ are scalars and } k > 0, l > 0)$$

Therefore, $k\mathbf{u} + l\mathbf{u}$ and $(k + l)\mathbf{u}$ are vectors equal in magnitude and direction. Thus

$$(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}\,.$$

Case 2. $k > 0, \quad l < 0, \quad |k| > |l|.$

Let $r = k + l$ and $s = -l$.

$$r > 0 \text{ and } s > 0\,.$$
$$(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}. \qquad \text{(Case 1)}$$
$$(k + l - l)\mathbf{u} = (k + l)\mathbf{u} - l\mathbf{u}.$$
$$k\mathbf{u} = (k + l)\mathbf{u} - l\mathbf{u}.$$

Therefore,

$$k\mathbf{u} + l\mathbf{u} = (k + l)\mathbf{u}\,.$$

Case 3. $k > 0, \quad l < 0, \quad |k| < |l|\,.$

Case 4. $k < 0, \quad l > 0, \quad |k| > |l|\,.$

Case 5. $k < 0, \quad l > 0, \quad |k| < |l|\,.$

Case 6. $k < 0, \quad l < 0\,.$

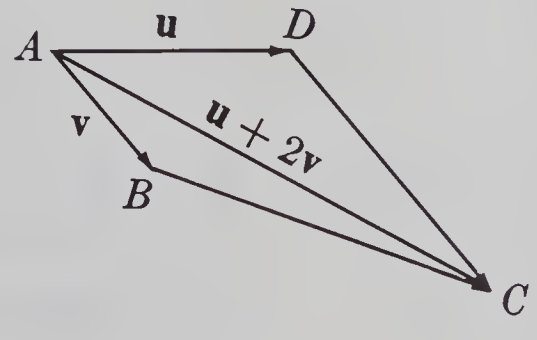
Case 7. $k = 0 \quad \text{or} \quad l = 0\,.$

Case 8. $k + l = 0\,.$

Case 9. $\mathbf{u} = \mathbf{0}\,.$

Demonstration of these cases is asked for in Exercise 1.7.

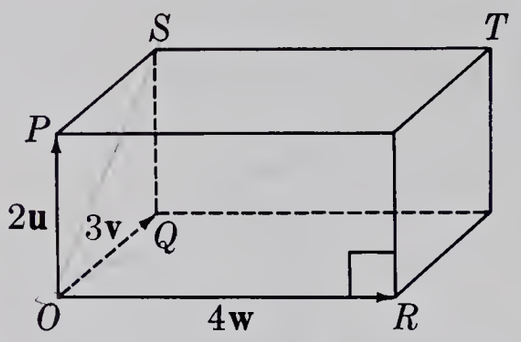
Example 1. $ABCD$ is a quadrilateral with $\overrightarrow{AD} = \mathbf{u}$, $\overrightarrow{AB} = \mathbf{v}$, $\overrightarrow{AC} = \mathbf{u} + 2\mathbf{v}$. Express $\overrightarrow{BC}$ and $\overrightarrow{DC}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.



Solution:

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{AC} - \overrightarrow{AB}\\ &= \mathbf{u} + 2\mathbf{v} - \mathbf{v}\\ &= \mathbf{u} + \mathbf{v}.\\ \overrightarrow{DC} &= \overrightarrow{AC} - \overrightarrow{AD}\\ &= \mathbf{u} + 2\mathbf{v} - \mathbf{u}\\ &= 2\mathbf{v}\,.\end{aligned}$$

Example 2. $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ are three vectors which are mutually perpendicular. If $\overrightarrow{OP} = 2\mathbf{u}$, $\overrightarrow{OQ} = 3\mathbf{v}$, and $\overrightarrow{OR} = 4\mathbf{w}$, express $\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$ in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. If $|\mathbf{u}| = 2$, $|\mathbf{v}| = 1$, and $|\mathbf{w}| = 3$, calculate $|\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}|$.



Solution:

$$\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} = 2\mathbf{u} + 3\mathbf{v} + 4\mathbf{w}.$$

$$\begin{aligned}
|\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}| &= |\overrightarrow{OS} + \overrightarrow{OR}| \\
&= |\overrightarrow{OT}| \\
&= \sqrt{|\overrightarrow{OS}|^2 + |\overrightarrow{OR}|^2} \\
&= \sqrt{|\overrightarrow{OP}|^2 + |\overrightarrow{OQ}|^2 + |\overrightarrow{OR}|^2} \\
&= \sqrt{|2\mathbf{u}|^2 + |3\mathbf{v}|^2 + |4\mathbf{w}|^2} \\
&= \sqrt{16 + 9 + 144} \\
&= 13.
\end{aligned}$$

EXERCISE 1.7

1. Demonstrate that
$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$
when
(a) $k < 0$, (b) $k = 0$, (c) $\mathbf{u}$ and $\mathbf{v}$ are collinear.

2. Use geometric vectors to demonstrate that
$$(-1)\mathbf{u} = -\mathbf{u}.$$

3. Show that
$$(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$$
when
(a) $k > 0, l < 0, |k| < |l|$, (b) $k < 0, l > 0, |k| > |l|$,
(c) $k < 0, l < 0$, (d) $k = 0$ or $l = 0$,
(e) $k + l = 0$, (f) $\mathbf{u} = \mathbf{0}$.

4. $PQRS$ is a quadrilateral with $\overrightarrow{PQ} = 2\mathbf{u}$, $\overrightarrow{QR} = 3\mathbf{v}$, $\overrightarrow{QS} = 3\mathbf{v} - 3\mathbf{u}$. Express $\overrightarrow{PS}$ and $\overrightarrow{RS}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

5. $ABCD$ is a quadrilateral and $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{CD} = 2\mathbf{v}$, $\overrightarrow{AC} = 3\mathbf{u} - \mathbf{v}$. Express $\overrightarrow{BD}$ and $\overrightarrow{BC}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

6. If $\mathbf{u} = k\mathbf{v}$, $k \neq 0$, prove that

$$\mathbf{v} = \frac{1}{k}\mathbf{u}.$$

7. If A, B, and C are three collinear points with B the midpoint of AC and O is any point not on the line AC, prove that

$$\overrightarrow{OA} + \overrightarrow{OC} = 2\,\overrightarrow{OB}.$$

(*Note*: $\overrightarrow{AB} = \overrightarrow{BC}$.)

8. If A, B, and C are three collinear points with B between A and C such that

$$AB : BC = 2 : 1,$$

prove that

$$\overrightarrow{OB} = \frac{2\,\overrightarrow{OC} + \overrightarrow{OA}}{3},$$

where O is any point not on the line AC.

9. $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are three mutually perpendicular vectors. $\overrightarrow{PQ} = 2\mathbf{u}$, $\overrightarrow{PR} = \mathbf{v}$, $\overrightarrow{PS} = 5\mathbf{w}$, $|\mathbf{u}| = 2$, $|\mathbf{v}| = 5$, and $|\mathbf{w}| = 4$. Express $\overrightarrow{PQ} + \overrightarrow{PR} + \overrightarrow{PS}$ in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and calculate $|\overrightarrow{PQ} + \overrightarrow{PR}|$, $|\overrightarrow{PR} + \overrightarrow{PS}|$, and $|\overrightarrow{PQ} + \overrightarrow{PR} + \overrightarrow{PS}|$.

10. A, B, C are three collinear points with B between A and C so that

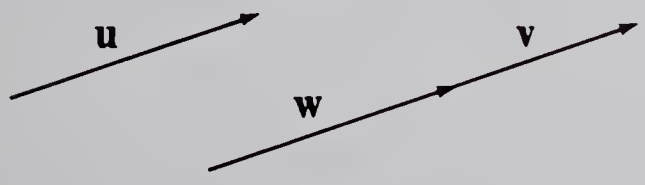
$$AB : BC = r : s.$$

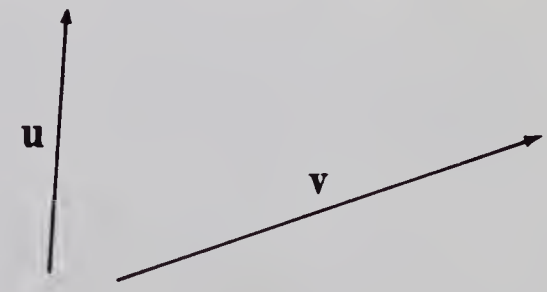
If O is any point not on the line AC, prove that

$$\overrightarrow{OB} = \frac{s\,\overrightarrow{OA} + r\,\overrightarrow{OC}}{s + r}.$$

1.8. Linear Combination of Vectors

If we consider two nonzero parallel vectors $\mathbf{u}$ and $\mathbf{v}$, it is possible to cut off on $\mathbf{v}$ a vector $\mathbf{w}$ equal to $\mathbf{u}$. $\mathbf{u}$ and $\mathbf{w}$ are equal vectors since they are equal in magnitude and direction. Also, for some scalar k, $\mathbf{v} = k\mathbf{w} = k\mathbf{u}$. $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$ and $\mathbf{u} = \frac{1}{k}\mathbf{v}$ is a scalar multiple of $\mathbf{v}$.





If the two vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, it is impossible to express one as a scalar multiple of the other. By the definition of multiplication by a scalar, the directions of $k\mathbf{u}$ and $\mathbf{u}$ are the same if $k > 0$ and opposite if $k < 0$. Hence, $\mathbf{v} \neq k\mathbf{u}$ for any scalar k, since the direction of $\mathbf{v}$ cannot be the same as the direction of $k\mathbf{u}$, although it is possible to find a vector $k\mathbf{u}$ such that

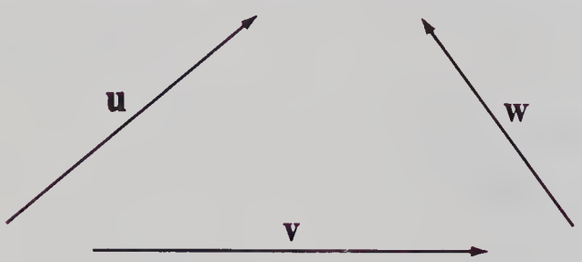
$$|\mathbf{v}| = |k\mathbf{u}|.$$

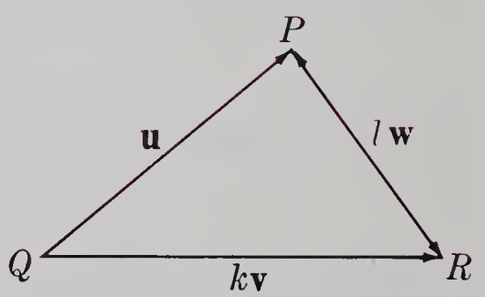
In the first diagram, $\mathbf{u}$ and $\mathbf{w}$ are equal vectors. In any discussion it is possible to replace one with the other. They are members of an equivalence class of vectors all equal in magnitude and direction. In practice we are always using one member of this class of vectors and we may call any such representative vector $\mathbf{u}$. Note that any two vectors in the equivalence class are either parallel to, or collinear with, $\mathbf{u}$. It follows that any vector $\mathbf{v} = k\mathbf{u}$ must also be parallel to or collinear with $\mathbf{u}$.

To simplify the situation we say that if $\mathbf{v} = k\mathbf{u}$, then $\mathbf{u}$ and $\mathbf{v}$ are collinear realizing that this means only that we can find a representative vector $\mathbf{v}$ which is collinear with $\mathbf{u}$.

DEFINITION. Two coplanar vectors are said to be collinear if and only if one is a scalar multiple of the other.

For example, if $\mathbf{v} = k\mathbf{u}$, then $\mathbf{v}$ and $\mathbf{u}$ are collinear.





If we consider three coplanar, noncollinear vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as shown, it is possible to construct $\triangle PQR$ in which $\overrightarrow{QP} = \mathbf{u}$, $\overrightarrow{QR} = k\mathbf{v}$, and $\overrightarrow{RP} = l\mathbf{w}$; that is, $\overrightarrow{QR}$ is parallel to $\mathbf{v}$ and $\overrightarrow{RP}$ is parallel to $\mathbf{w}$.

$$\overrightarrow{QP} = \overrightarrow{QR} + \overrightarrow{RP}.$$

Therefore,

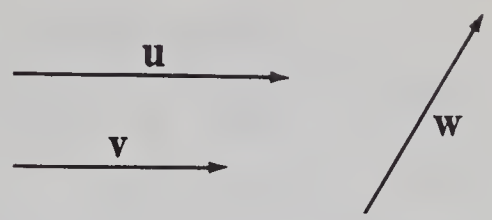
$$\mathbf{u} = k\mathbf{v} + l\mathbf{w}.$$

This construction is possible for any three coplanar, noncollinear vectors, and we say that $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Also in this case, since $k \neq 0$, $l \neq 0$,

$$\mathbf{v} = \frac{1}{k}\mathbf{u} - \frac{l}{k}\mathbf{w}, \text{ and } \mathbf{v} \text{ is a linear combination of } \mathbf{u} \text{ and } \mathbf{w},$$

$$\mathbf{w} = \frac{1}{l}\mathbf{u} - \frac{k}{l}\mathbf{v}, \text{ and } \mathbf{w} \text{ is a linear combination of } \mathbf{u} \text{ and } \mathbf{v}.$$

If **u** and **v** are collinear (or parallel) vectors, and **w** is not parallel to **u** and **v**, then **w** is not a linear combination of **u** and **v**, but **u** is a linear combination of **v** and **w**, and **v** is a linear combination of **u** and **w**.



Since **u** and **v** are collinear, then for some scalar k, $\mathbf{u} = k\mathbf{v}$, and

$$\begin{aligned}\mathbf{u} &= k\mathbf{v} + \mathbf{0} \\ &= k\mathbf{v} + 0\mathbf{w}.\end{aligned}$$

Therefore,

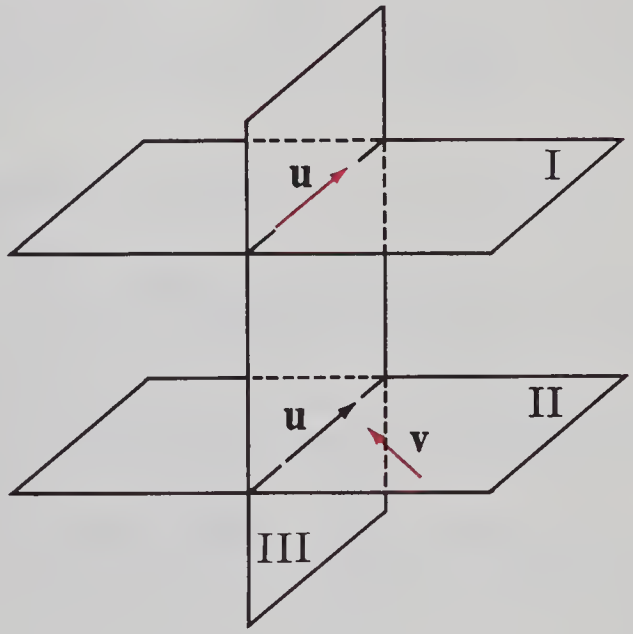
$$\mathbf{v} = \frac{1}{k}\mathbf{u} + 0\mathbf{w}.$$

u is a linear combination of **v** and **w**, **v** is a linear combination of **u** and **w**, but **w** is not a linear combination of **u** and **v**.

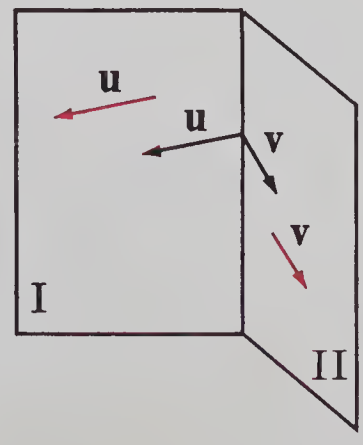
DEFINITION. Three vectors are coplanar if and only if at least one is a linear combination of the other two.

The word "coplanar" can include parallel planes in the same way as the word collinear included parallel lines when discussing vectors.

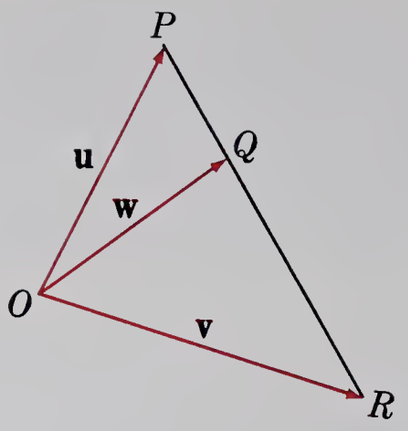
In the diagram, I and II are parallel planes with **u** a vector in I, and **v** a vector in II. A plane III intersects plane I in the line through **u** and plane III is perpendicular to I. Plane III is also perpendicular to II and intersects II in a line parallel to its line of intersection with I. Hence a vector equal to **u** in magnitude and direction can be drawn in plane II. Thus **u** and **v** are coplanar. It should again be emphasized that, in this way, any two vectors are coplanar, although two line segments are not necessarily coplanar.



If **u** and **v** are vectors in two nonparallel planes I and II, representative vectors **u** and **v** can be drawn with initial points on the line of intersection of the planes. In this case, however, the plane in which both **u** and **v** lie is a plane different from I and II. The initial point common to both **u** and **v** is the point of intersection of the three planes. We suggest that these cases be examined by using sheets of paper to represent the planes.

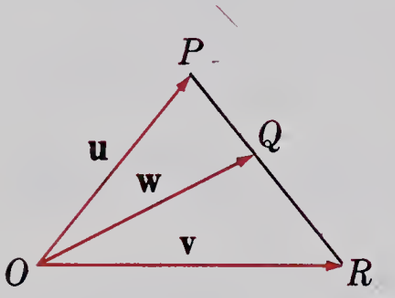


In the diagram P, Q, and R are collinear and are the endpoints of $\overrightarrow{OP} = \mathbf{u}$, $\overrightarrow{OQ} = \mathbf{w}$, $\overrightarrow{OR} = \mathbf{v}$. The vectors have a common initial point O. These vectors are known as the position vectors of P, Q, and R relative to the reference point O. Since a line and a point not on the line determine a plane, the vectors are all coplanar, and any one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is a linear combination of the other two.

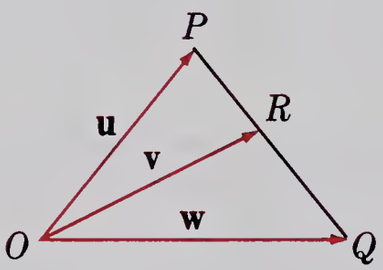


Theorem. If $\mathbf{u} = \overrightarrow{OP}$, $\mathbf{w} = \overrightarrow{OQ}$, and $\mathbf{v} = \overrightarrow{OR}$ are three vectors with P, Q, and R collinear, and $\overrightarrow{PQ} = k\,\overrightarrow{QR}$, then

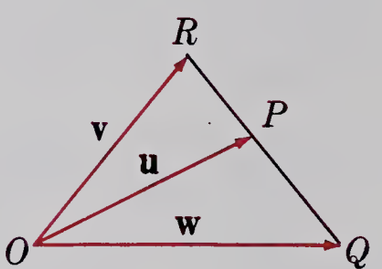
$$\mathbf{w} = \frac{k\mathbf{v} + \mathbf{u}}{k+1}.$$



(1)



(2)



(3)

In case (1), $k > 0$ and Q is between P and R.

In case (2), $|k| > 1$, since $|\overrightarrow{PQ}| > |\overrightarrow{QR}|$, and k is negative because $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are opposite in direction; thus $k < -1$.

In case (3), $|k| < 1$, since $|\overrightarrow{PQ}| < |\overrightarrow{QR}|$, and k is negative because $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are opposite in direction; thus $-1 < k < 0$.

Note that $\overrightarrow{PQ} = k\overrightarrow{QR}$ may be written in the form $PQ : QR = k : 1$, which shows that Q divides the line segment PR in the ratio $k : 1$.

Proof:

P, Q, R are collinear

and

$$\overrightarrow{PQ} = k\,\overrightarrow{QR}.$$

Therefore,

$$\overrightarrow{PO} + \overrightarrow{OQ} = k\,(\overrightarrow{QO} + \overrightarrow{OR}).$$

Hence,

$$-\mathbf{u} + \mathbf{w} = k(-\mathbf{w} + \mathbf{v}).$$

$$\mathbf{w} + k\mathbf{w} = k\mathbf{v} + \mathbf{u}.$$

Therefore,

$$\mathbf{w} = \frac{k\mathbf{v} + \mathbf{u}}{k+1}.$$

Note that if Q is the midpoint of PR so that $k = 1$, then

$$\mathbf{w} = \frac{\mathbf{v} + \mathbf{u}}{2} = \tfrac{1}{2}\mathbf{v} + \tfrac{1}{2}\mathbf{u}\,.$$

Since all the steps in the solution are reversible, the converse theorem may be proved by simply reversing the steps in the solution.

Converse Theorem. If $\mathbf{u} = \overrightarrow{OP}$, $\mathbf{w} = \overrightarrow{OQ}$, and $\mathbf{v} = \overrightarrow{OR}$ are such that

$$\mathbf{w} = \frac{k\mathbf{v} + \mathbf{u}}{k + 1}\,,$$

then P, Q, and R are collinear, and Q divides the line segment PR in the ratio $k : 1$.

The proof is required in the exercises.

The definitions of parallel or collinear vectors and coplanar vectors together with the theorem just proved enable us to prove many theorems of deductive geometry by vector methods. We should note that if

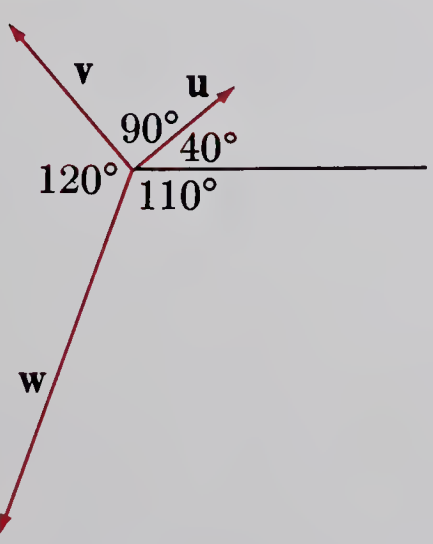
$$\overrightarrow{AB} = k\overrightarrow{CD}\,,$$

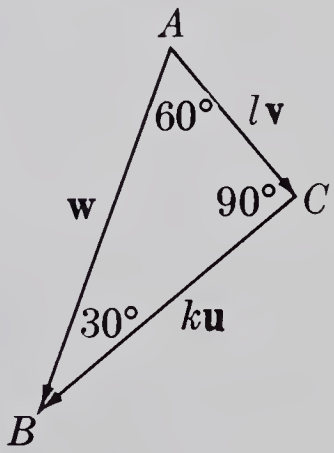
then the lines containing $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are either collinear or parallel. If these lines have no point in common, then

$$AB \parallel CD\,.$$

If they have one point in common, then AB and CD are collinear.

Example 1. Three coplanar vectors **u**, **v**, and **w** make angles of 40°, 130°, and 250°, respectively, with the horizontal. If $|\mathbf{u}| = 4$, $|\mathbf{v}| = 6$, and $|\mathbf{w}| = 12$, express **w** as a linear combination of **u** and **v**.





Solution: Let

$$\mathbf{w} = l\mathbf{v} + k\mathbf{u}\,.$$

In $\triangle ABC$, $AB = 12$, $AC = 6|l|$, $BC = 4|k|$, $\angle ABC = 30°$, $\angle ACB = 90°$, $\angle BAC = 60°$.

$$AC = AB \sin 30°.$$

Thus,

$$6|l| = 12 \times \tfrac{1}{2}.$$
$$|l| = 1.$$
$$BC = AB \cos 30°.$$
$$4|k| = \tfrac{12\sqrt{3}}{2}.$$

Therefore,

$$|k| = \tfrac{3\sqrt{3}}{2}.$$

From the diagram, $k < 0$, $l < 0$; therefore $l = -1$, $k = -\tfrac{3\sqrt{3}}{2}$, and

$$\mathbf{w} = -\mathbf{v} - \tfrac{3\sqrt{3}}{2}\mathbf{u}.$$

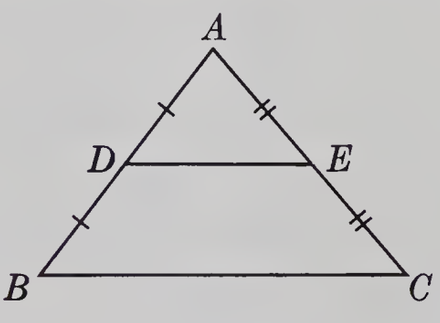
Note also that

$$\tfrac{3\sqrt{3}}{2}\mathbf{u} = -\mathbf{v} - \mathbf{w}.$$

Therefore,

$$\mathbf{u} = -\tfrac{2}{3\sqrt{3}}\mathbf{v} - \tfrac{2}{3\sqrt{3}}\mathbf{w}$$
$$= -\tfrac{2\sqrt{3}}{9}(\mathbf{v} + \mathbf{w}).$$

Example 2. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal in length to one half of it.



Given: $\triangle ABC$ with D the midpoint of AB and E the midpoint of AC.

Required: To prove $DE \parallel BC$ and

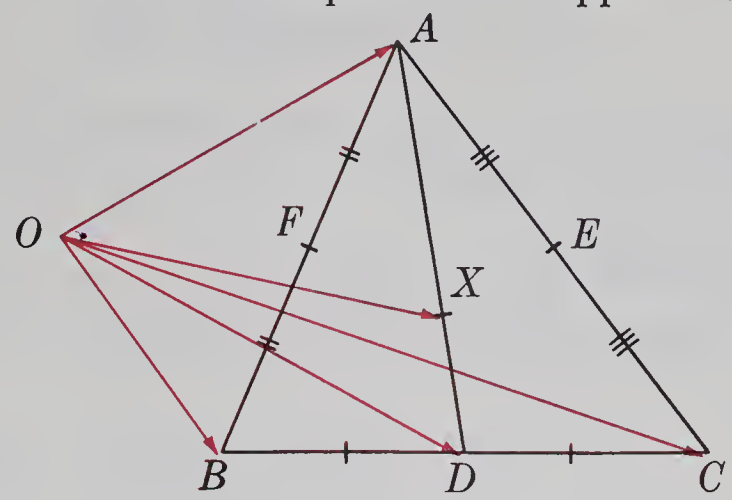
$$DE = \tfrac{1}{2}BC.$$

Proof:

$$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$$
$$= \tfrac{1}{2}\overrightarrow{BA} + \tfrac{1}{2}\overrightarrow{AC}$$
$$= \tfrac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$$
$$= \tfrac{1}{2}\overrightarrow{BC}.$$

Therefore, $DE \parallel BC$ and $DE = \tfrac{1}{2}BC$.

Example 3. Prove that the medians of a triangle intersect at a point two thirds of the distance from each vertex to the midpoint of the opposite side.



Given: $\triangle ABC$.

Required: To prove that the medians of $\triangle ABC$ intersect at a point two thirds of the distance from each vertex to the midpoint of the opposite side.

Proof: Let D, E, F be midpoints of BC, AC, and AB, respectively. Let X be the point on AD such that

$$AX : XD = 2 : 1 .$$

Let O be any point not necessarily in the plane of ABC. Therefore,

$$\overrightarrow{OX} = \frac{2\overrightarrow{OD} + \overrightarrow{OA}}{2+1}$$
$$= \tfrac{2}{3}\overrightarrow{OD} + \tfrac{1}{3}\overrightarrow{OA} ,$$

but

$$\overrightarrow{OD} = \tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC} .$$

Therefore,

$$\overrightarrow{OX} = \tfrac{2}{3}(\tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC}) + \tfrac{1}{3}\overrightarrow{OA}$$
$$= \tfrac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) .$$

Similarly, if Y is the point on BE such that

$$BY : YE = 2 : 1 ,$$

then

$$\overrightarrow{OY} = \tfrac{1}{3}(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OA}) ,$$

and if Z is the point on CF such that

$$CZ : ZE = 2 : 1 ,$$

then

$$\overrightarrow{OZ} = \tfrac{1}{3}(\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OB}) .$$

Therefore,

$$\overrightarrow{OX} = \overrightarrow{OY} = \overrightarrow{OZ} ,$$

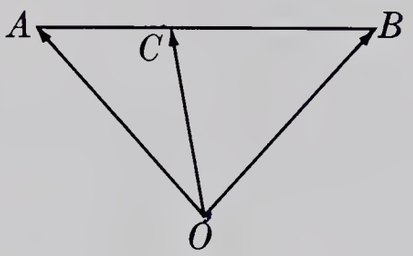
and X, Y, Z are the same point. Therefore, the medians intersect at a point two thirds of the distance from each vertex to the midpoint of the opposite side.

The final part may be shortened by noting that the result

$$\overrightarrow{OX} = \tfrac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$$

is completely symmetrical in A, B, and C and is independent of the median used. Therefore, X is on all three medians.

Example 4. If $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are such that A, B, and C are collinear and $AB : BC = -5 : 3$, express $\overrightarrow{OB}$ as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.



Solution:

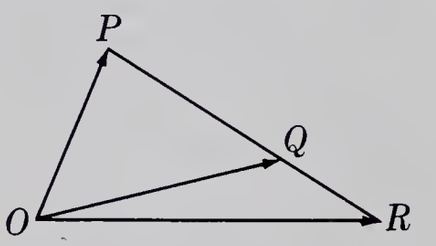
$$AB : BC = -\tfrac{5}{3} : 1 .$$

$$\overrightarrow{OB} = \frac{-\frac{5}{3}\overrightarrow{OC} + \overrightarrow{OA}}{-\frac{5}{3} + 1} .$$

$$= \frac{-5\,\overrightarrow{OC} + 3\,\overrightarrow{OA}}{-5 + 3}$$

$$= \tfrac{5}{2}\,\overrightarrow{OC} - \tfrac{3}{2}\,\overrightarrow{OA} .$$

Example 5. If $\overrightarrow{OP} = 3\,\overrightarrow{OQ} - 2\,\overrightarrow{OR}$, prove that P, Q, and R are collinear, and find the value of the ratio $PQ : QR$.



Solution:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \overrightarrow{OQ} - 3\,\overrightarrow{OQ} + 2\,\overrightarrow{OR}$$
$$= 2(\overrightarrow{OR} - \overrightarrow{OQ}).$$
$$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$$
$$= \overrightarrow{OR} - \overrightarrow{OQ}.$$

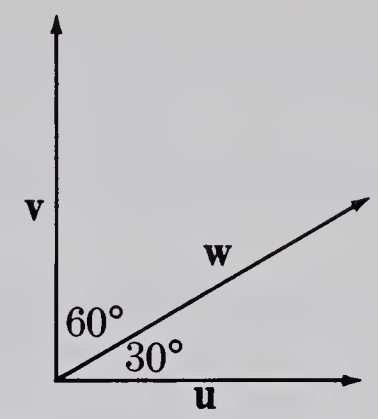
Thus,

$$\overrightarrow{PQ} = 2\,\overrightarrow{QR}.$$

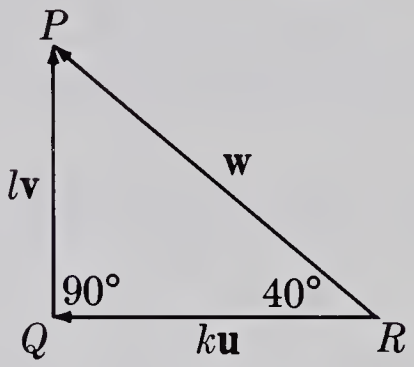
Therefore, P, Q and R are collinear and $PQ : QR = 2 : 1$.

EXERCISE 1.8

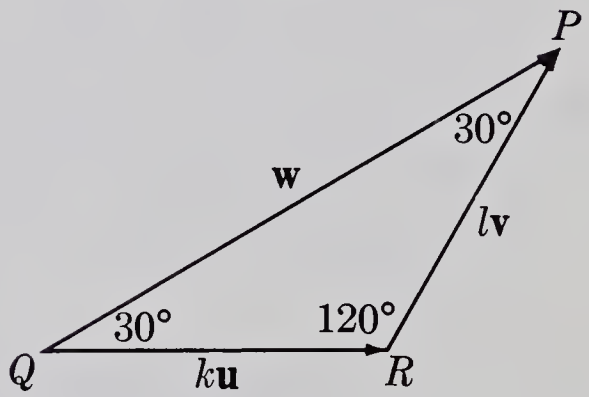
1. In the diagram, **u**, **v**, **w** represent three coplanar vectors. If $|\mathbf{u}| = 10$, $|\mathbf{v}| = 12$, and $|\mathbf{w}| = 12$, express **w** as a linear combination of **u** and **v**. Also express **v** as a linear combination of **u** and **w**.



2. In the diagram, $\overrightarrow{RP} = \mathbf{w}$, $\overrightarrow{RQ} = k\mathbf{u}$, $\overrightarrow{QP} = l\mathbf{v}$ with $k > 0$, $l > 0$. If $|\mathbf{u}| = 4$, $|\mathbf{v}| = 5$, and $|\mathbf{w}| = 16$, find the values of k and l correct to 2 decimal places.



3. In the diagram, $\overrightarrow{QP} = \mathbf{w}$, $\overrightarrow{QR} = k\mathbf{u}$, $\overrightarrow{RP} = l\mathbf{v}$ with $k > 0$, $l > 0$. If $|\mathbf{u}| = 2$, $|\mathbf{v}| = 4$, and $|\mathbf{w}| = 10$, express **w** as a linear combination of **u** and **v**.



4. $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are such that A, B, and C are collinear with
$$AB : BC = 3 : 2 .$$
Express $\overrightarrow{OB}$ as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

5. If $\overrightarrow{OP} = \mathbf{u}$, $\overrightarrow{OQ} = \mathbf{v}$, $\overrightarrow{OR} = \mathbf{w}$ and P, Q, and R are collinear with
$$PQ : QR = -1 : 4,$$
express **v** as a linear combination of **u** and **w**.

6. If $\overrightarrow{OQ} = \frac{2}{5}\overrightarrow{OP} + \frac{3}{5}\overrightarrow{OR}$, prove that P, Q, and R are collinear and that $PQ : QR = 3 : 2$.

7. If $\mathbf{u} = \overrightarrow{OP}$, $\mathbf{w} = \overrightarrow{OQ}$, and $\mathbf{v} = \overrightarrow{OR}$ are such that
$$\mathbf{w} = \frac{k\mathbf{v} + \mathbf{u}}{k + 1},$$
prove that P, Q, and R are collinear and that $PQ : QR = k : 1$.

Use vector methods to prove each of the following.

8. Prove that if one pair of opposite sides of a quadrilateral are equal and parallel, then the other pair of opposite sides are also equal and parallel.

9. Prove that the diagonals of parallelogram $ABCD$ bisect each other. (Hint: Let E be the midpoint of AC and F the midpoint of BD. Prove $\overrightarrow{AE} = \overrightarrow{AF}$.)

10. In $\triangle ABC$, D and E are points on AB and AC, respectively, such that $AD : DB = AE : EC$. Prove that $DE \parallel BC$ and that $DE : BC = AD : AB$.

11. AC and BD are two line segments which bisect each other at E. Prove $ABCD$ is a parallelogram.

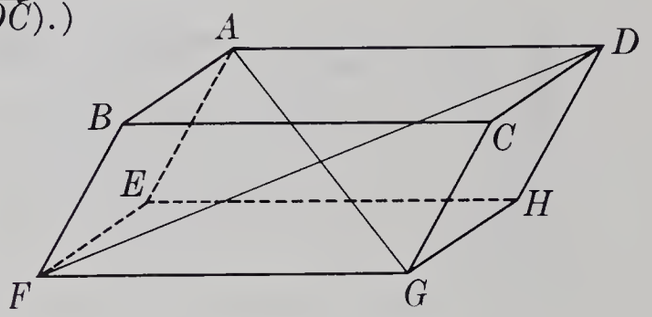
12. $ABCD$ is a parallelogram with E the midpoint of AD. Prove that AC and BE trisect each other (see hint to question (9)).

13. In $\triangle ABC$, D is the midpoint of AB and $DE \parallel BC$ meets AC at E. Prove that E is the midpoint of AC.

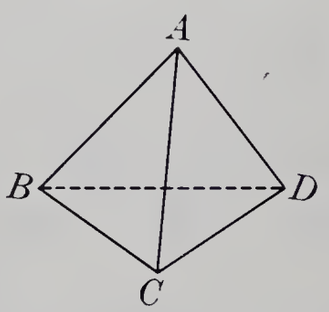
14. $ABCD$ is a parallelogram with E the midpoint of AD and F the midpoint of BC. Prove that BE and DF trisect AC.

15. P, Q, R, and S are the centroids of triangles ABC, ABD, DEF, and CEF, respectively. Prove that $PQRS$ is a parallelogram. (Let O be any point and note that $\overrightarrow{OP} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.)

16. The diagram shows a parallelopiped (opposite faces are parallel and congruent parallelograms). Prove that the diagonals AG and DF bisect each other.



17. Prove that the three line segments joining the midpoints of pairs of opposite edges of a tetrahedron bisect each other. A tetrahedron is a pyramid with a triangular base as shown in the diagram. AB and CD are opposite edges, as are AD and BC, and AC and BD.



Chapter Summary

Definitions

Skew lines are lines in space which do not intersect and are not parallel.

A line l is perpendicular to a plane I at a point N if it is perpendicular to every line in the plane passing through N.

The measure of a dihedral angle is the measure of the plane angle formed by the perpendiculars in each plane at a point on the line of intersection of the planes.

Two vectors are equal if and only if they have the same magnitude and the same direction.

If $\mathbf{u}$ and $\mathbf{v}$ are two vectors represented by $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, respectively, so that the endpoint of the line segment PQ is the initial point of the line segment QR, then the sum of $\mathbf{u}$ and $\mathbf{v}$ is represented by $\overrightarrow{PR}$.

The negative of a vector $\mathbf{u}$ is a vector equal in magnitude to $\mathbf{u}$ but opposite in direction, and is denoted by $-\mathbf{u}$.

The resultant force of two or more forces is that single force that would have the same effect as the given forces.

The equilibrant force of two or more forces acting on a body is that single force which, when applied to the body, maintains it in a state of equilibrium.

If $\mathbf{u}$ is a nonzero vector and k a nonzero real number (scalar), the vector $k\mathbf{u}$ is defined by the following rules.

(1) $|k\mathbf{u}| = |k|\,|\mathbf{u}|$.

(2) The directions of $k\mathbf{u}$ and $\mathbf{u}$ are the same if $k > 0$ and opposite if $k < 0$.

Also

$$k\mathbf{0} = \mathbf{0} \quad \text{and} \quad 0\mathbf{u} = \mathbf{0}\,.$$

Two coplanar vectors are collinear if one is a scalar multiple of the other.

Three vectors are coplanar if at least one is a linear combination of the other two.

Algebraic Properties of Vectors

For the following properties, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors.

(1) $\mathbf{u} + \mathbf{v}$ is a vector.	Closure property
(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.	Commutative property
(3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.	Associative property
(4) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.	Identity property (zero vector)
(5) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$.	Inverse property

Properties of Multiplication by a Scalar

For the following properties, $\mathbf{u}$ and $\mathbf{v}$ are vectors and k and l are real numbers.

(1) $k\mathbf{u}$ is a vector.

(2) $k\mathbf{u} = \mathbf{u}k$.

(3) If $k = l$ and $\mathbf{u} = \mathbf{v}$, then $k\mathbf{u} = l\mathbf{v}$.

(4) $(kl)\mathbf{u} = k(l\mathbf{u})$.

(5) $1\mathbf{u} = \mathbf{u}$.

(6) $(-1)\mathbf{u} = -\mathbf{u}$.

(7) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

(8) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$.

(9) $|k\mathbf{u}| = |k|\,|\mathbf{u}|$.

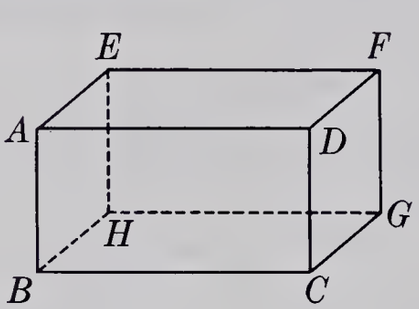
Theorem.

If $\mathbf{u} = \overrightarrow{OP}$, $\mathbf{w} = \overrightarrow{OQ}$, and $\mathbf{v} = \overrightarrow{OR}$ are three vectors with P, Q, and R collinear and $\overrightarrow{PQ} = k\overrightarrow{QR}$, then

$$\mathbf{w} = \frac{k\mathbf{v} + \mathbf{u}}{k + 1}.$$

REVIEW EXERCISE 1

1. $PQRST$ is a pentagon. State a single vector equal to each of the following.
   (a) $\overrightarrow{PQ} + \overrightarrow{QR}$ (b) $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RT}$ (c) $\overrightarrow{QR} + \overrightarrow{RT}$
   (d) $\overrightarrow{ST} - \overrightarrow{PT}$ (e) $\overrightarrow{PQ} + \overrightarrow{QR} - \overrightarrow{TR}$ (f) $\overrightarrow{PR} - (\overrightarrow{PT} - \overrightarrow{ST})$



2. The diagram above represents a rectangular prism. State a single vector equal to each of the following.
   (a) $\overrightarrow{AB} + \overrightarrow{BG}$ (b) $\overrightarrow{EH} + \overrightarrow{HG}$ (c) $\overrightarrow{AB} + \overrightarrow{HG}$
   (d) $\overrightarrow{BC} + \overrightarrow{DF} + \overrightarrow{GH}$ (e) $\overrightarrow{FG} - \overrightarrow{BH}$ (f) $\overrightarrow{CG} - \overrightarrow{EH}$
   (g) $\overrightarrow{AD} - \overrightarrow{HC} - \overrightarrow{GH}$ (h) $\overrightarrow{DF} + \overrightarrow{GC}$

3. Two vectors $\mathbf{u}$ and $\mathbf{v}$ have a common initial point and form an angle of 140°. If $|\mathbf{u}| = 4$ and $|\mathbf{v}| = 8$, calculate, to the nearest integer, the value of $|\mathbf{u} + \mathbf{v}|$ and, to the nearest degree, the size of the angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{v}$.

4. Simplify $3\mathbf{u} + 2\mathbf{v} - 2(\mathbf{v} - \mathbf{u}) + (-3\mathbf{u})$.

5. Two forces of 40 pounds and 60 pounds act on a body at an angle of 60° to each other. Find the magnitude and direction of the resultant force, to the nearest pound and the nearest degree.

6. Two forces of 50 grams and 80 grams act on a body at an angle of 130° to each other. Find the magnitude and direction of the equilibrant, to the nearest gram and nearest degree.

7. Two equal forces acting at an angle of 80° to each other have a resultant of 60 pounds. Calculate the magnitude of the two forces, to the nearest pound.

8. A light aircraft is flying at a speed of 160 m.p.h. on a bearing of 140°. A 30 m.p.h. wind is blowing from a bearing of 290°. Calculate the ground speed and direction of flight of the aircraft, to the nearest m.p.h. and the nearest degree.

9. A ship's navigator wishes to set a course for a port which is 300 nautical miles distant from his present position on a bearing of 220°. The ship's speed is 18 knots and a current of 6 knots is travelling from a bearing of 100°. What course should he set and what will be the actual speed of the ship, to the nearest degree and nearest knot?

10. A force of 40 pounds acts at an angle of 50° to the horizontal. Calculate the horizontal and vertical components of the force, to the nearest pound.

11. Resolve a force of 65 pounds into two rectangular components so that the magnitude of the components are in the ratio 5 : 12. Calculate the angle between the 65-pound force and the larger component. Give the results to the nearest unit.

12. An aircraft is climbing at an angle of 20° to the horizontal and at a speed of 120 m.p.h. What is its rate of climb in feet per minute and its horizontal ground speed in miles per hour, each to nearest unit.

13. Three coplanar vectors **u**, **v**, and **w** make angles of 30°, 60°, and 120°, respectively, with the horizontal. If $|\mathbf{u}| = 5, |\mathbf{v}| = 6, |\mathbf{w}| = 8$, express **v** as a linear combination of **u** and **w**.

14. Three coplanar vectors **u**, **v**, and **w** make angles of 60°, 180°, and −30° with the horizontal. If $|\mathbf{u}| = 3, |\mathbf{v}| = 2, |\mathbf{w}| = 5$, express **u** as a linear combination of **v** and **w**.

15. In $\triangle ABC$, E is the midpoint of AC and BE is produced to D so that $ED = BE$. Using vector methods, prove $AB \parallel CD$ and $AB = CD$.

16. In $\triangle PQR$, S is a point on QR such that
$$QS : SR = 3 : 2 ,$$
and T is a point on PS such that
$$PT : TS = -5 : 2 .$$
Using vector methods, prove $PQ \parallel RT$.

17. $ABCD$ is a quadrilateral with P, Q, R, and S the midpoints of AB, BC, CD, and DA, respectively. Using vector methods, prove that $PQRS$ is a parallelogram.

18. $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OB} = \mathbf{v}$, $\overrightarrow{OC} = \mathbf{w}$, and A, B, C are collinear with $AB : BC = 2 : 1$. Express **v** as a linear combination of **u** and **w**.

19. $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ are such that P, Q, R are collinear with $PQ : QR = -2 : 5$. Express $\overrightarrow{OQ}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

20. If $\overrightarrow{OQ} = \frac{3}{7}\overrightarrow{OP} + \frac{4}{7}\overrightarrow{OR}$, prove that P, Q, R are collinear and that
$$PQ : QR = 4 : 3.$$

21. If $\overrightarrow{OQ} = \frac{s}{r}\overrightarrow{OP} + \frac{r-s}{r}\overrightarrow{OR}$, prove that P, Q, R are collinear and that $PQ : QR = (r - s) : s$.

22. In a tetrahedron $ABCD$, P, Q, R, and S are the centroids of faces ABC, BCD, CDA, and ABD, respectively. Prove that DP, AQ, BR, and CS intersect at X so that

$$DX : XP = AX : XQ = BX : XR = CX : XS = 3 : 1.$$

(Hint: take as the origin of the position vectors any point not on the faces of the tetrahedron.)

23. Show that the point X found in question (22) is also the point of intersection of the line segments joining the midpoints of the opposite edges of the tetrahedron (see question (17), Exercise 1.8).

24. Prove that the diagonals of a cube intersect in a point and show that the line segments joining the midpoints of the opposite edges intersect in this point.

25. State and prove results for a rectangular prism similar to those in question (24) for the cube. Do similar results hold for the general parallelopiped?

Chapter 2

ALGEBRAIC VECTORS

2.1. Graphical Representation of Triplets of Real Numbers

A pair (x, y) of real numbers may be represented geometrically by a point in a plane using a Cartesian co-ordinate system with a frame of reference consisting of two lines (the x- and y-axes) intersecting at right angles in a point (the origin). The real numbers, x and y, are the co-ordinates of the particular point.

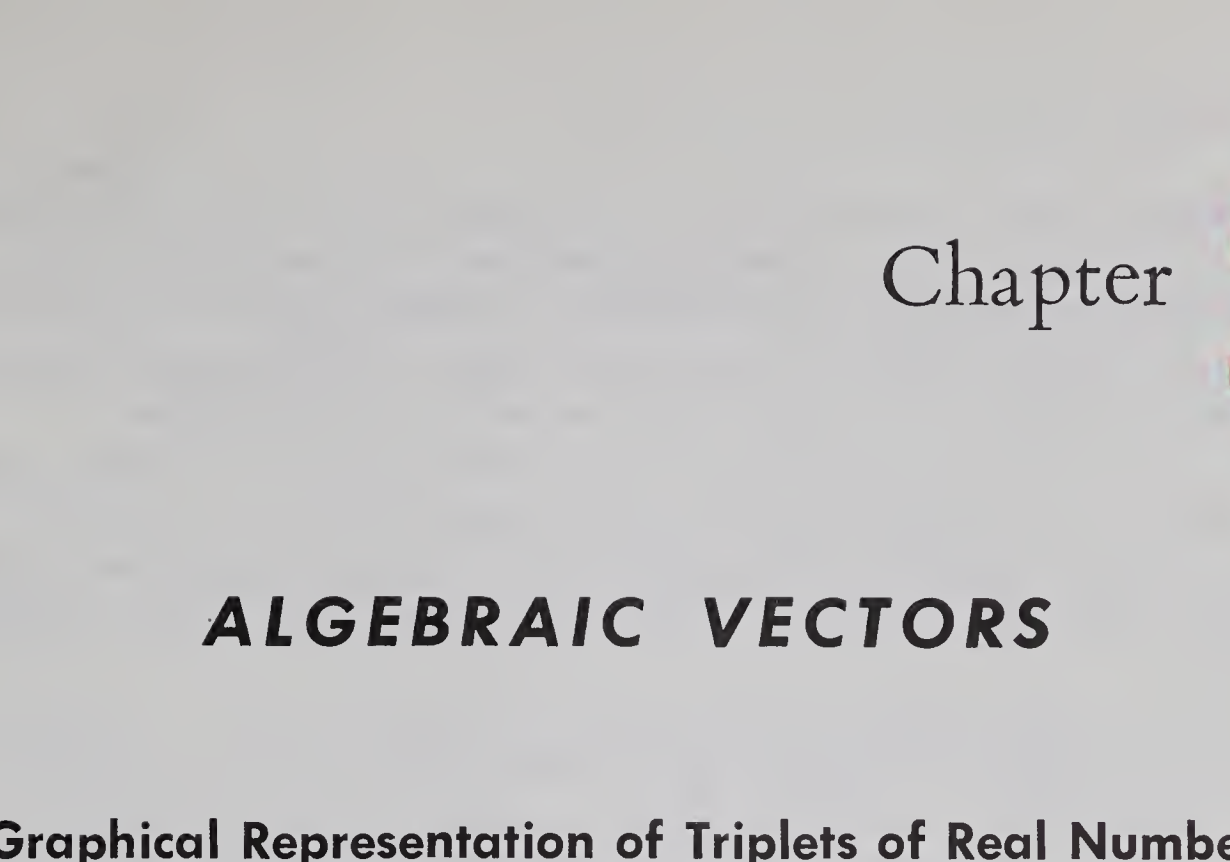


Figure 2.1

The points of a particular plane may all be represented by pairs of real numbers. Therefore, the consideration of triplets of real numbers would suggest going *outside* a particular plane. Indeed, triplets (x, y, z) of real numbers are represented by points in space. In this case, our frame of reference consists of three mutually

perpendicular number lines (the x-, y-, and z-co-ordinate axes) intersecting in a point O (the origin). Such a frame of reference is shown in Figure 2.1. The three axes are at right angles to each other but, of course, cannot be drawn as such in the plane of the page; they are drawn at an angle of 120° to each other and the student must visualize the x- and y-axes as coming out of the page.

The x-, y-, and z-axes are three number lines and, by convention, the solid rays represent the positive directions of these number lines.

Note that the co-ordinate axes, taken in pairs, form three mutually perpendicular planes called the co-ordinate planes, namely, the xy-plane, the yz-plane, and the xz-plane.

To determine the point represented by the triplet (x, y, z) of real numbers, we start at the origin O, move x units along the x-axis, then y units in the xy-plane parallel to the y-axis, and finally z units from that point parallel to the z-axis. Distances are measured to the same scale along all three axes. Figure 2.2 shows the representation of some typical points.

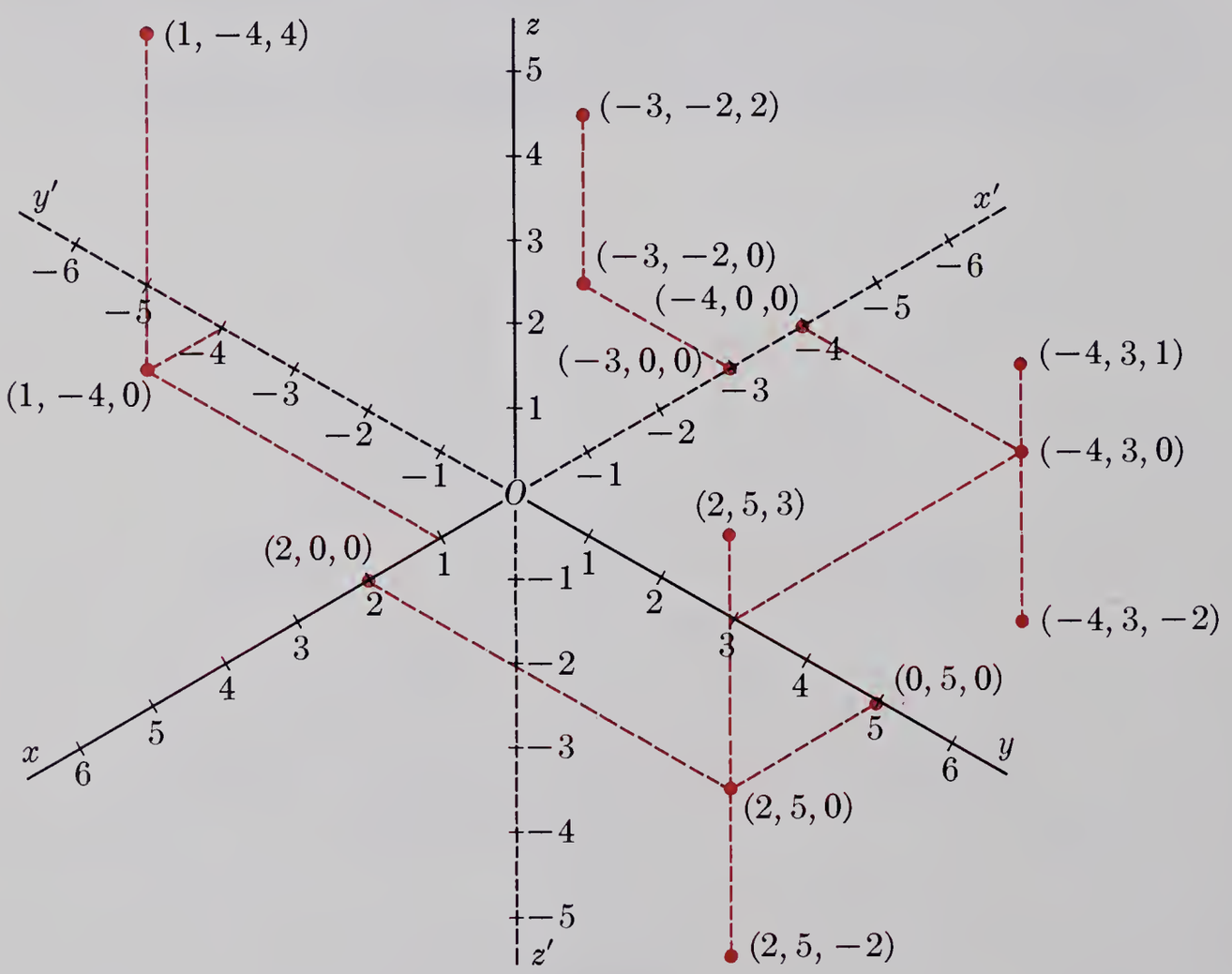


Figure 2.2

Conversely, a point in space determines a triplet of real numbers with respect to such a co-ordinate system, namely, real numbers x (perpendicular distance of the point from the yz-plane), y (distance of the point from the xz-plane), and z

(distance of the point from the xy-plane). Then there is a one-to-one correspondence between the set of points in space and the set of ordered triplets (x, y, z) of real numbers, just as there is a one-to-one correspondence between the set of points in a plane and the set of ordered pairs (x, y) of real numbers.

EXERCISE 2.1

1. Plot the following triplets of real numbers on a suitably drawn co-ordinate system.

$(1, 1, 1)$	$(1, 3, -4)$	$(0, -5, 2)$	$(1, -3, -6)$
$(-4, 3, 5)$	$(-3, 6, -2)$	$(-3, -2, 0)$	$(3, -4, -4)$
$(-2, -3, -4)$	$(-5, 1, 6)$	$(4, -3, 0)$	$(-1, 0, -6)$
$(-5, 0, 0)$	$(3, 0, 0)$	$(0, -4, 0)$	$(0, 0, 7)$.

2. The three co-ordinate axes in space divide space into eight regions called octants. The region bounded by the boldfaced rays is called the first octant (the others are not numbered). What characterizes points in the first octant? How would you refer to any particular octant (without attempting to number the octants)?

3. What characterizes points on the yz-plane? on the xz-plane? on the xy-plane? What, then, are the equations of these planes?

4. What characterizes points on the x-axis? on the y-axis? on the z-axis?

5. A line in space does not have a single equation as does a line in a plane, but rather a pair of equations. What are the pairs of equations for the x-axis? the y-axis? the z-axis?

2.2. Two- and Three-Dimensional Vectors

In the preceding section, ordered pairs and triplets of real numbers were given geometric interpretations as points in a plane and in space, respectively. In the present section, we define a (real) two-dimensional vector to be an ordered pair of real numbers and a (real) three-dimensional vector to be an ordered triplet of real numbers, and we shall consider some algebraic properties of these mathematical objects.

Example 1. $(0, 0)$, $(3, 6)$, $(-\frac{1}{2}, 5)$, $(\pi, -\pi)$ are two dimensional vectors; $(0, 0, 0)$, $(-1, 3, 6)$, $(-\frac{1}{2}, 5, 0)$, $(\pi, 3.14, \frac{22}{7})$ are three-dimensional vectors.

The real numbers involved in the symbols in the example above are called the components of the vector. We normally use a bold-faced letter (in print) to refer to a vector: thus $\mathbf{u} = (1, 2)$, $\mathbf{v} = (3, 2, 5)$, etc. In writing, we may use either $\underline{u} = (1, 2)$ or $\vec{u} = (1, 2)$.

Let us now consider the set V_2 of all ordered pairs, and the set V_3 of all ordered triplets of real numbers. When we consider sets of mathematical objects, it is important that we know when two of the elements are equal. We have the following definitions in V_2 and V_3.

DEFINITION. $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

$$(a, b, c) = (d, e, f) \text{ if and only if } a = d,\ b = e,\ \text{and } c = f.$$

Two vectors are equal if and only if they have the same dimension and corresponding components are equal.

Example 2. $(2, -1) \neq (3, 2, -1)$. (Why?)
$(2, 3, 1) \neq (2, 3, -1)$. (Why?)
$(a, 3) = (-2, b)$ if and only if $a = -2$ and $b = 3$.
$(3, -1, 2) \neq (-3, a, b)$ for any values of a and b.

Vectors may be added according to the following definition.

DEFINITION.
$$(a, b) + (c, d) = (a + c, b + d).$$
$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f).$$

Notice that addition is defined only for equidimensional vectors and that equidimensional vectors are added by the addition of corresponding components. Obviously both V_2 and V_3 are closed under this operation of addition.

Example 3. $(-1, 5) + (3, 4) = (2, 9)$.
$(6, -4, 2) + (-1, 0, 5) = (5, -4, 7)$.
$(2, 3) + (2, 3, 0)$ is not defined.

A vector may be multiplied by a real number according to the following definition.

DEFINITION.
$$t(a, b) = (ta, tb).$$
$$t(a, b, c) = (ta, tb, tc).$$

According to this definition, a vector is multiplied by a real number by the multiplication of each component by that real number. The product of a two- (three-) dimensional vector and a real number is a two- (three-) dimensional vector; V_2 and V_3 are closed under multiplication by real numbers.

Example 4. $3(5, -1) = (15, -3)$.
$-2(6, -1, 4) = (-12, 2, -8)$.

It is easy to verify that $(0, 0)$ and $(0, 0, 0)$ are identities under addition in V_2 and V_3, respectively; we call them zero vectors and often refer to them simply as $\mathbf{0}$.

Note that

$$(a, b) + (-a, -b) = (0, 0) ,$$

and

$$(a, b, c) + (-a, -b, -c) = (0, 0, 0) .$$

We call $(-a, -b)$ the additive inverse or negative of (a, b), and $(-a, -b, -c)$ the additive inverse or negative of (a, b, c). Then, as usual, the subtraction of one vector from a second is defined as the addition of the negative of that vector to the second.

DEFINITION.
$$\begin{aligned}(a, b) - (c, d) &= (a, b) + (-c, -d) \\ &= (a - c, b - d) .\end{aligned}$$

$$\begin{aligned}(a, b, c) - (d, e, f) &= (a, b, c) + (-d, -e, -f) \\ &= (a - d, b - e, c - f) .\end{aligned}$$

Example 5. The negative of $(3, -7, 5)$ is $(-3, 7, -5)$.

Check: $(3, -7, 5) + (-3, 7, -5) = (0, 0, 0) .$

Example 6.
$$\begin{aligned}(4, -7, 3) - (3, 2, -4) &= (4, -7, 3) + (-3, -2, 4) \\ &= (1, -9, 7) .\end{aligned}$$

As usual, the second step in the subtraction operation is normally omitted and we simply subtract corresponding components of the vectors.

Note that

$$(-1)(a, b, c) = (-a, -b, -c) ,$$

so that

$$(-1)(a, b, c)$$

is the negative of (a, b, c).

EXERCISE 2.2

1. Express the following as single vectors.

(a) $(2, 3) + (4, 6)$
(b) $(3, 2) + (-5, 1)$
(c) $(4, 7) + (0, 0)$
(d) $(3, -1, 4) + (2, 0, 1)$
(e) $(3, 0, -5) + (-3, -2, 1)$
(f) $(0, 0, -5) + (4, 0, 0)$
(g) $5(2, 8)$
(h) $-3(4, 7)$
(i) $0(0, 0)$
(j) $3(-2, -1, 4)$
(k) $-2(-4, -1, 3)$
(l) $\frac{1}{2}(4, 2, -6)$
(m) $(4, 7) - (4, 7)$
(n) $(4, 7) - (3, 2)$
(o) $(-4, 7) - (-5, -3)$
(p) $(3, 2, 1) - (1, 2, 3)$
(q) $(-4, 1, 2) - (3, -1, -2)$
(r) $(-2, -1, 6) - (3, -1, 0)$

2. Show that
   (a) $(-1, 3) + (4, 5) = (4, 5) + (-1, 3)$,
   (b) $(2, 1, 6) + (-3, 2, 5) = (-3, 2, 5) + (2, 1, 6)$.

3. Use the vectors
   (a) $(3, 1)$, $(2, -4)$, $(5, -3)$,
   (b) $(4, 2, 1)$, $(3, -2, 0)$, $(5, -2, -1)$,
   to demonstrate that addition of vectors is associative.

4. Evaluate $3[(2, 1, 5) + (-2, 4, 6)]$ in two ways to illustrate that multiplication by a real number is distributive over the addition of vectors.

5. Find a and b if
   $$3(a, 1) - 2(2, b) = (2, 1).$$

6. Find a, b, and c if
   $$2(a, -1, 4) - 3(-4, b, 6) - \tfrac{1}{2}(4, -2, c) = (0, 0, 0).$$

7. Find x, y, and z if
   $$3[(x, 1, -1) + (3, -2, z)] - (4, 3, -1) = 6(-3, -y, 5).$$

8. Find p, q, and r if
   $$(p, -6, 4) - 2(q, p, 3) + (5, q, r) = (0, 0, 0).$$

9. Prove that addition of two- and three-dimensional vectors is commutative.

10. Prove that addition of two- and three-dimensional vectors is associative.

11. Prove that multiplication by a real number is distributive over addition of two- and three-dimensional vectors.

2.3. Identification of Algebraic and Geometric Vectors

In Chapter 1, we discussed physical vectors represented geometrically by directed line segments in a plane (2-space) or in a space of three dimensions (3-space). It was pointed out that the characteristics of such representations, called geometric vectors, were length and direction. Two geometric vectors are equal if they are line segments having the same length and the same direction, and equal geometric vectors are normally interchangeable in a geometric discussion.

Consider the ordered pair (2, 3) of real numbers, that is, the algebraic vector (2, 3) in V_2. Moving $+2$ units in the x direction from the origin O of a plane Cartesian co-ordinate system, and then $+3$ units in the y direction brings us to a point P in the plane whose co-ordinates are (2, 3); O and P are the initial and terminal points, respectively, of a directed line segment or vector $\overrightarrow{OP}$ (Figure 2.3). Indeed, associated with every nonzero algebraic vector (a, b) in V_2 there is a geometric vector with initial point O and terminal point P with co-ordinates (a, b).

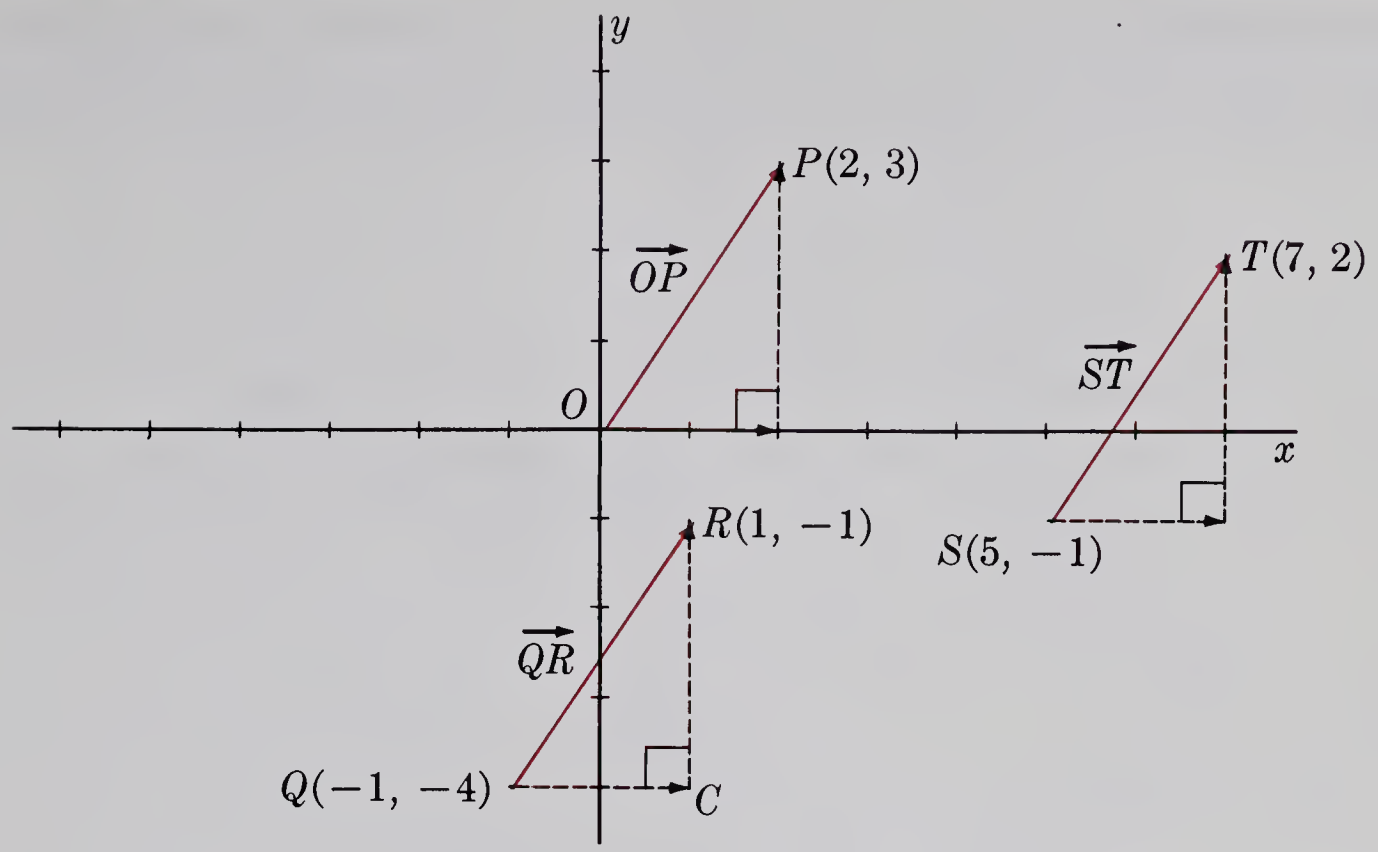


Figure 2.3

Now choose an arbitrary point, say $Q(-1, -4)$, and move $+2$ units in the x and $+3$ units in the y direction; this procedure determines the point $R(1, -1)$ and a vector $\overrightarrow{QR}$. Similarly, starting at $S(5, -1)$ determines the point $T(7, 2)$ and the vector $\overrightarrow{ST}$ (Figure 2.3). Obviously, $\overrightarrow{OP} = \overrightarrow{QR} = \overrightarrow{ST}$, since $\overrightarrow{OP}$, $\overrightarrow{QR}$, and $\overrightarrow{ST}$ are line segments with the same length and the same direction (OAP, QCR, and SBT are congruent right-angled triangles). We may also say that $\overrightarrow{OP}$, $\overrightarrow{QR}$, $\overrightarrow{ST}$ are vectors of the same equivalence class.

Thus, each nonzero algebraic vector in V_2 has an infinite number of equivalent geometric representatives in the plane. In particular, there is a one-to-one correspondence between the nonzero vectors of V_2 and the geometric vectors *with initial point at the origin* in a fixed plane.

Note the double interpretation that we now have for an ordered pair (a, b) of real numbers

(i) as an algebraic vector of V_2,

(ii) as the co-ordinates of a point P in the plane that is the terminal point of a geometric vector with initial point at the origin.

The geometric vector $\overrightarrow{OP}$ from the origin O to the point P with co-ordinates (a, b) can thus be identified with the algebraic vector (a, b).

In the same way, an ordered triplet (a, b, c) of real numbers can be considered as an algebraic vector of V_3, or as the co-ordinates of a point P in space that determines, with the origin O, the geometric vector $\overrightarrow{OP}$. For example, $(3, -2, 2)$ is an algebraic vector in V_3; also, the point P with co-ordinates $(3, -2, 2)$ determines the geometric vector $\overrightarrow{OP}$ (Figure 2.4). Again, starting at any point Q in

space and moving $+3$ units in the x direction, -2 units in the y direction, and $+2$ units in the z direction determines a point R and a vector $\overrightarrow{QR}$ such that $\overrightarrow{QR} = \overrightarrow{OP}$ (OAB and QCD are congruent right triangles, as are OBP and QDR).

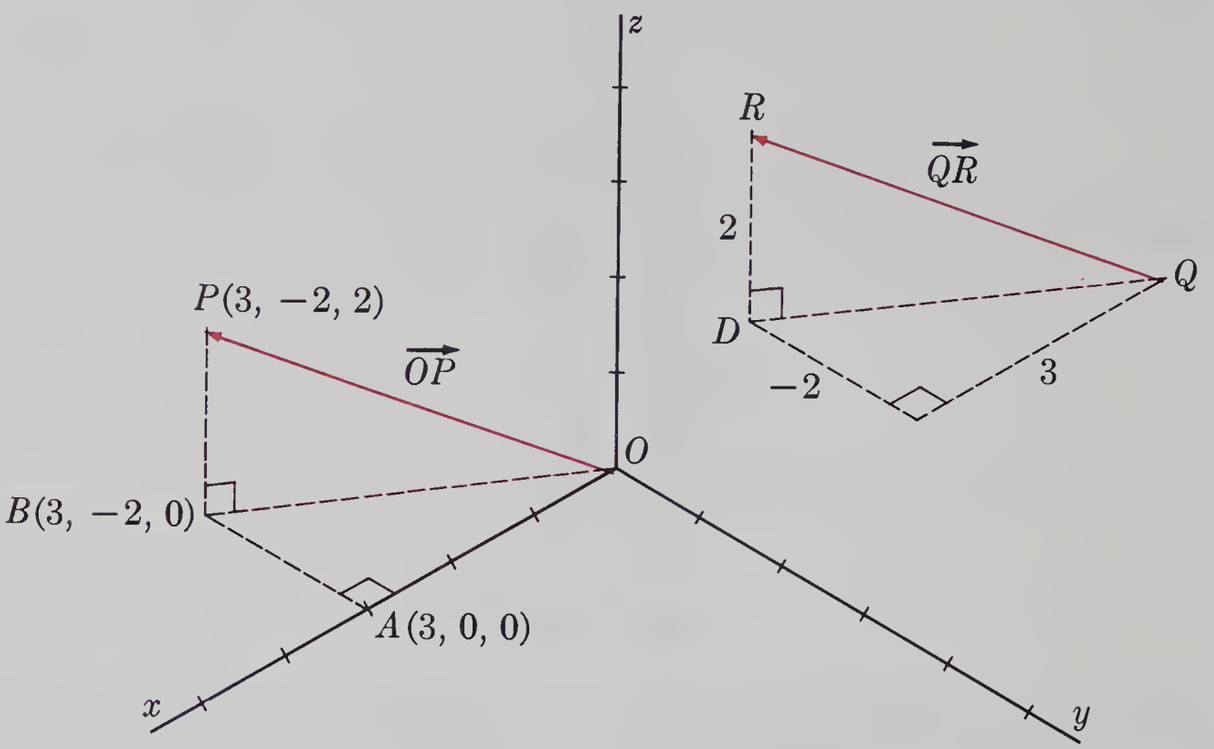


Figure 2.4

Thus, there is a one-to-one correspondence between nonzero algebraic vectors of V_3 and geometric vectors in 3-space *with initial point at the origin*, but each nonzero algebraic vector in V_3 has an infinite number of equivalent geometric representatives in 3-space.

EXERCISE 2.3

1. Draw representatives in 2-space with initial point at
   (i) the origin, (ii) (3, 4), (iii) (−1, 5), (iv) (−3, −5), (v) (4, −3),
   for the following vectors in V_2.
   (a) (1, 1) (b) (2, 3) (c) (−3, 4) (d) (−4, 3)
   (e) (3, 5) (f) (2, 0) (g) (−1, −2) (h) (0, −3)

2. Draw representatives in 3-space with initial point at
   (i) the origin, (ii) (3, 3, 4), (iii) (−2, 1, 5), (iv) (3, −2, 4), (v) (−2, −2, −2),
   for the following vectors in V_3.
   (a) (1, 1, 1) (b) (2, 3, 1)
   (c) (−3, 4, 2) (d) (2, 2, 2)
   (e) (−4, 3, 0) (f) (2, 0, 0)
   (g) (−1, −2, −2) (h) (0, 0, −3)

3. What algebraic vectors in V_2 correspond to geometric vectors $\overrightarrow{PQ}$ if P and Q are the following points?
   (a) $P(0, 0), Q(2, 5)$ (b) $P(0, 0), Q(-3, 1)$
   (c) $P(0, 0), Q(-5, -2)$ (d) $P(2, 5), Q(0, 0)$
   (e) $P(-3, 1), Q(0, 0)$ (f) $P(-5, -2), Q(0, 0)$
   (g) $P(3, 4), Q(5, 7)$ (h) $P(3, -4), Q(-2, 5)$
   (i) $P(-3, -1), Q(-5, -5)$ (j) $P(-4, 2), Q(-7, -3)$

4. What point P in the plane is the terminal point of a geometric vector $\overrightarrow{OP}$ that is equivalent to $\overrightarrow{RS}$ for the following points R and S?
   (a) $R(4, 3), S(2, 1)$ (b) $R(6, 3), S(-5, 3)$
   (c) $R(-3, 2), S(5, 7)$ (d) $R(-1, -3), S(-7, -2)$

5. What algebraic vectors in V_3 correspond to geometric vectors $\overrightarrow{PQ}$ if P and Q are the following points?
   (a) $P(0, 0, 0), Q(3, -1, 4)$ (b) $P(3, 2, 4), Q(0, 0, 0)$
   (c) $P(-2, -1, -1), Q(0, 0, 0)$ (d) $P(3, 3, 1), Q(7, 2, 1)$
   (e) $P(-2, 5, 1), Q(7, 3, 2)$ (f) $P(-1, 3, -4), Q(-1, -5, -4)$

6. What point P in 3-space is the terminal point of a geometric vector $\overrightarrow{OP}$ that is equivalent to $\overrightarrow{RS}$ for the following points R and S?
   (a) $R(2, -3, 1), S(-4, -2, 1)$ (b) $R(3, -1, -2), S(5, 4, 3)$
   (c) $R(-1, -2, -5), S(-3, -4, -1)$ (d) $R(-1, 2, 1), S(6, 0, -2)$

7. What is the connection between the algebraic vectors corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{QP}$?

8. Compare geometric vectors representing (a, b, c) and $(-a, -b, -c)$.

9. Is there a geometric vector representing the algebraic vector $(0, 0, 0)$?

10. When the geometric vector has the indicated property, what can be said about the corresponding algebraic vector (a, b, c) in each of the following cases?
   (a) parallel to the x-axis (b) parallel to the y-axis
   (c) parallel to the z-axis (d) perpendicular to the x-axis
   (e) perpendicular to the y-axis (f) perpendicular to the z-axis

2.4. Geometric Interpretation of $\mathbf{u} + \mathbf{v}$ and $t\mathbf{u}$

The parallelogram law for the addition of geometric vectors in two and three dimensions was established in Chapter 1. In this section, we shall show that this addition of geometric vectors corresponds to the addition of algebraic vectors as defined in Section 2.2.

Example 1. Draw geometric vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ corresponding to the algebraic vectors (5, 2) and (2, 4). Find the sum $\overrightarrow{OR}$ of these vectors by using the parallelogram law, and show that this vector represents the algebraic sum of (5, 2) and (2, 4).

Solution: Figure 2.5 shows vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ corresponding to algebraic vectors (5, 2) and (2, 4). If R is such that $OPRQ$ is a parallelogram, then $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$ by the parallelogram law.

Draw $QA \perp x$-axis, $RB \perp x$-axis, and $PB \perp RB$. Then triangles OAQ and PBR are congruent (why?) so that PB and BR have lengths 2 and 4 units, respectively, and R has co-ordinates (7, 6).

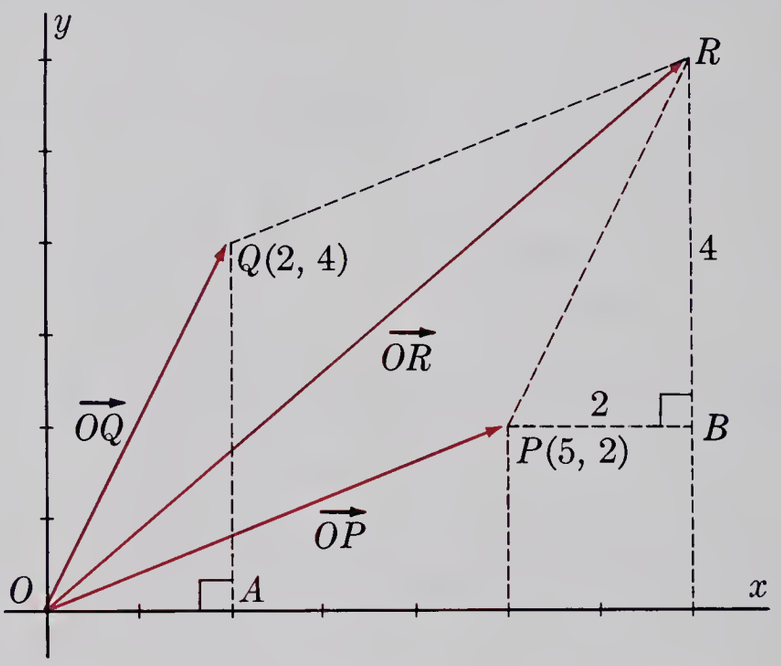


Figure 2.5

But, by the definition of algebraic vectors,

$$(2, 4) + (5, 2) = (7, 6)$$

and so $\overrightarrow{OR}$ is the geometric vector corresponding to the algebraic vector (7, 6).

Example 2. Draw vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ corresponding to the algebraic vectors (4, 1), (−3, 4), and (4, 1) + (−3, 4). Show that $OPRQ$ is a parallelogram.

Solution: Figure 2.6 shows vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$; R has co-ordinates (1, 5).

$$OP \text{ has slope } \tfrac{1}{4}.$$

$$QR \text{ has slope } \frac{5-4}{1-(-3)} = \frac{1}{4}.$$

Therefore,

$$OP \parallel QR.$$

$$OQ \text{ has slope } \frac{4}{-3}.$$

$$PR \text{ has slope } \frac{5-1}{1-4} = \frac{4}{-3}.$$

Therefore,

$$OQ \parallel PR .$$

Therefore, $OPRQ$ is a parallelogram and, by the parallelogram law, $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$.

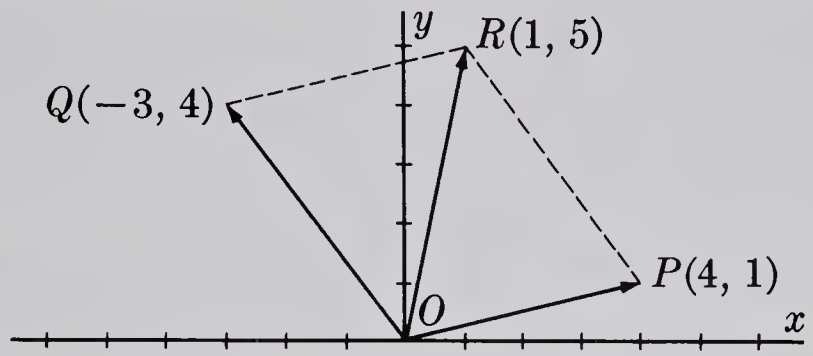


Figure 2.6

In Chapter 1, it was pointed out that the geometric vectors $\mathbf{u}$ and $k\mathbf{u}$ were collinear or parallel. The next example verifies that the vectors (a, b) and $k(a, b)$ are represented by parallel geometric vectors.

Example 3. Draw vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ corresponding to the algebraic vectors (2, 1) and (6, 3); show that O, P, and Q are collinear.

Solution: Figure 2.7 shows the required vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$; P has co-ordinates (2, 1) and Q has co-ordinates (6, 3).

$$OP \text{ has slope } \tfrac{1}{2}.$$

$$OQ \text{ has slope } \tfrac{3}{6} = \tfrac{1}{2}.$$

Therefore,

$$OP \parallel OQ ,$$

and so O, P, and Q are collinear.

We may conclude from this result that *any* geometric representative of $\mathbf{u} = (2, 1)$ is parallel to *any* geometric representative of $3\mathbf{u} = 3(2, 1)$.

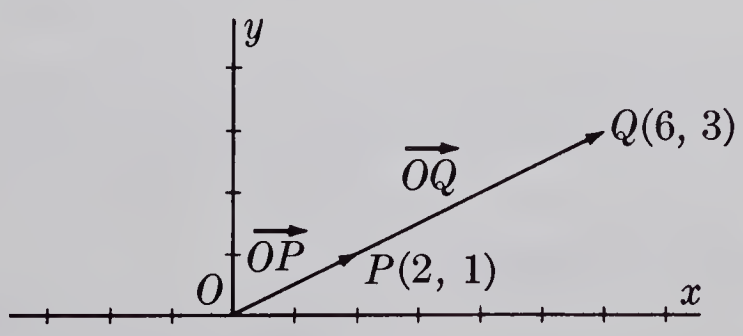


Figure 2.7

The examples of this section have illustrated in the two-dimensional case

(i) that addition of algebraic vectors corresponds to the addition of geometric vectors using the parallelogram law,

(ii) that multiplication of an algebraic vector by a real number produces a vector whose geometric representatives are parallel to the representatives of the original vector; in other words, (a, b) and $k(a, b)$ are represented by parallel vectors.

These two facts are also true in the three-dimensional case. Their verification will be easier when we have further information about lines in space at our disposal. (See question (10), Exercise 4.6.)

Of course, the vectors $\mathbf{v} = (a_1, b_1, c_1)$ and $\mathbf{w} = (a_2, b_2, c_2)$ are represented by the line segments OP and OQ joining the origin to the points $P(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$. These three points O, P, and Q determine a plane and the result above on addition of vectors then applies in this plane.

EXERCISE 2.4

1. Draw geometric vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ corresponding to the following pairs of algebraic vectors. Find the sum $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$ using the parallelogram law and show that $\overrightarrow{OR}$ represents the sum of the given pair of vectors.
   (a) $(3, 2), (6, 1)$ (b) $(-4, 3), (2, 3)$
   (c) $(2, -3), (-3, 1)$ (d) $(4, -3), (-2, -3)$
   (e) $(-2, -3), (5, 3)$ (f) $(-3, -1), (-4, 3)$

2. Draw vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ corresponding to the following pairs of algebraic vectors and their sums, respectively, and show in each case that $OPRQ$ is a parallelogram.
   (a) $(1, 4)$ and $(3, 5)$ (b) $(-2, 6)$ and $(4, -2)$
   (c) $(5, 2)$ and $(-3, -2)$ (d) $(-3, -6)$ and $(-1, 4)$

3. Find geometrically the sum of the following vectors; check your results algebraically.
   (a) $(1, 1), (3, -2), (2, 1)$ (b) $(-3, 1), (2, -4), (-3, -4)$

4. Draw three representatives of the vector $(-1, 3)$ and of the vector $(-2, 6)$, and show that the first set is parallel to the second.

5. Show that the geometric vectors representing $(-1, 4)$ and $2(-1, 4)$ are parallel.

6. Show that the geometric vector representing $(1, 3)$ with initial point at $(2, 0)$ is parallel to the geometric vector representing $-2(1, 3)$ with initial point at $(0, 3)$.

7. Find geometrically the following vectors.
   (a) $2(3, 1) + 3(1, 1)$ (b) $2(-1, 2) + 3(2, -1)$
   (c) $-1(2, -3) + 3(-1, 2)$ (d) $-2(1, 3) - 3(2, -1) + \frac{1}{4}(4, -6)$

8. Prove that geometric vectors representing (a, b) and $k(a, b)$ are parallel.
9. Prove that if $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ represent (a, b), (c, d), and $(a + c, b + d)$, respectively, then $OPRQ$ is a parallelogram.
10. Show algebraically and geometrically that if $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 3)$, then $2(\mathbf{u} + \mathbf{v}) = 2\mathbf{u} + 2\mathbf{v}$.
11. Show algebraically and geometrically that $-3(\mathbf{u} + \mathbf{v}) = -3\mathbf{u} - 3\mathbf{v}$ if $\mathbf{u} = (3, -2)$ and $\mathbf{v} = (-1, 4)$.
12. Show algebraically and geometrically that if $\mathbf{u} = (2, -1)$ and $\mathbf{v} = (-1, 3)$, $2(\mathbf{u} - \mathbf{v}) = 2\mathbf{u} - 2\mathbf{v}$.

2.5. Lines Through the Origin • Collinear Vectors

We have already seen that the geometric vectors corresponding to the algebraic vectors $\mathbf{u}$ and $k\mathbf{u}$, where k is a real number, are parallel line segments. If we restrict our geometric vectors to those with initial point at the origin, then $\mathbf{u}$ and $k\mathbf{u}$ lie along the same line through the origin, and hence are collinear.

Thus, for a given nonzero vector $\mathbf{u}$, the set

$$\{k\mathbf{u} \mid k \in Re\}$$

is represented by the set of all line segments OP joining the origin to the points P of a certain line; that is, there is a one-to-one correspondence between the set

$$\{k\mathbf{u} \mid k \in Re\}$$

and the set of points on a particular line through the origin (either in 2- or 3-space). In this sense, we say that the set of all scalar multiples of a fixed nonzero vector *determines* or *is* a line through the origin.

Example 1. $\{k(1, 2) \mid k \in Re\}$ determines the line in 2-space through the origin and the point with co-ordinates $(1, 2)$.

Example 2. $\{k(-1, 3, 2) \mid k \in Re\}$ determines the line in 3-space through the origin and the point with co-ordinates $(-1, 3, 2)$.

Two nonzero vectors are said to be collinear if their geometric representatives with initial point at the origin lie along the same line. Therefore, the condition that two nonzero vectors be collinear is that each vector be a scalar multiple of the other.

$\mathbf{u}$ and $\mathbf{v}$ are collinear if and only if $\mathbf{u} = k\mathbf{v}$.

Obviously, $\mathbf{0} = 0\mathbf{u}$ and $\mathbf{0}$ is considered to be collinear with every vector.

Example 3. Are the vectors

(a) $(1, 3, -2)$ and $(4, 12, -8)$, (b) $(2, 3)$ and $(-4, -7)$ collinear?

Solution:

(a) $(4, 12, -8) = 4(1, 3, -2)$, and $(1, 3, -2) = \frac{1}{4}(4, 12, -8)$. These two vectors are collinear.

(*b*) There is no real number k such that $(-4, -7) = k(2, 3)$. For there to be agreement in the first components, k would have to have the value -2, but $(-2)(3) \neq -7$. The two vectors are not collinear.

EXERCISE 2.5

1. State the sets of vectors determining lines parallel to the following vectors.
   (a) $(1, 3)$ (b) $(-3, 4)$
   (c) $(3, 5, -2)$ (d) $(-6, -1, 5)$
   (e) $(1, -2, 0)$ (f) $(-2, 4, 6)$

2. State the sets of vectors determining lines parallel to $\overrightarrow{PQ}$ if P and Q are the following points.
   (a) $P(2, 3)$, $Q(5, 7)$ (b) $P(-3, 1)$, $Q(5, -2)$
   (c) $P(4, -4)$, $Q(-3, -2)$ (d) $P(-1, -5)$, $Q(-6, -3)$
   (e) $P(2, 3, 1)$, $Q(-3, 1, 5)$ (f) $P(-1, 3, -4)$, $Q(3, 0, -2)$

3. Are the following pairs of vectors collinear?
   (a) $(4, -2)$, $(-2, 1)$ (b) $(-4, 2)$, $(-2, -1)$
   (c) $(1, 3, -2)$, $(\frac{1}{2}, \frac{3}{2}, -1)$ (d) $(3, -6, 5)$, $(\frac{3}{2}, 3, \frac{5}{2})$
   (e) $(1, -2, 2)$, $(-4, 8, -8)$ (f) $(3, -1, -5)$, $(6, -3, -10)$

4. Find values of the variable involved (if possible) so that the following pairs of vectors are collinear.
   (a) $(3, -2)$, $(6, k)$ $k = -4$ (b) $(-3, 1)$, $(-6, -k)$
   (c) $(1, 5, -3)$, $(-2, k, 6)$ (d) $(4, -1, 1)$, $(\frac{1}{2}, k, l)$
   (e) $(3, -1, 4)$, $(k, -3, l)$ (f) $(0, 2, -4)$, $(k, 4, l)$

5. The following vectors determine lines through the origin. Give the coordinates of two points on each line.
   (a) $k(2, 3)$ (b) $(3k, -k)$
   (c) $(2k, 0, -k)$ (d) $(k, 2k, -3k)$
   (e) $(a, -a, \sqrt{2}a)$ (f) $(\pi p, -2\pi p, 0)$

2.6. Planes Through the Origin • Coplanar Vectors

Suppose that **u** and **v** are two noncollinear vectors. Their geometric representatives with initial point at the origin determine two distinct lines, which intersect in the origin. But two intersecting lines determine a plane and so two noncollinear vectors determine a plane passing through the origin.

The sum $\mathbf{u} + \mathbf{v}$ of the noncollinear vectors **u** and **v** is represented geometrically by the diagonal of the parallelogram determined by **u** and **v**. This diagonal is a geometric vector in the plane determined by **u** and **v**. It follows that, for any real numbers k and l, the vector $k\mathbf{u} + l\mathbf{v}$ has a geometric representative having its initial point at the origin and lying in the plane determined by **u** and **v**. Indeed, it is obvious from the geometry (Figure 2.8) that, for given noncollinear vectors **u** and **v**, $k\mathbf{u} + l\mathbf{v}$ determines a point P in the plane determined by **u** and **v** (the point P such that $\overrightarrow{OP} = k\mathbf{u} + l\mathbf{v}$), and conversely, corresponding to every point P in the plane determined by **u** and **v**, there are two real numbers k and l such that $\overrightarrow{OP} = k\mathbf{u} + l\mathbf{v}$. (Note that we have not specified whether **u** and **v** are both two- or three-dimensional vectors; they can be either, provided they both have the same dimensions.)

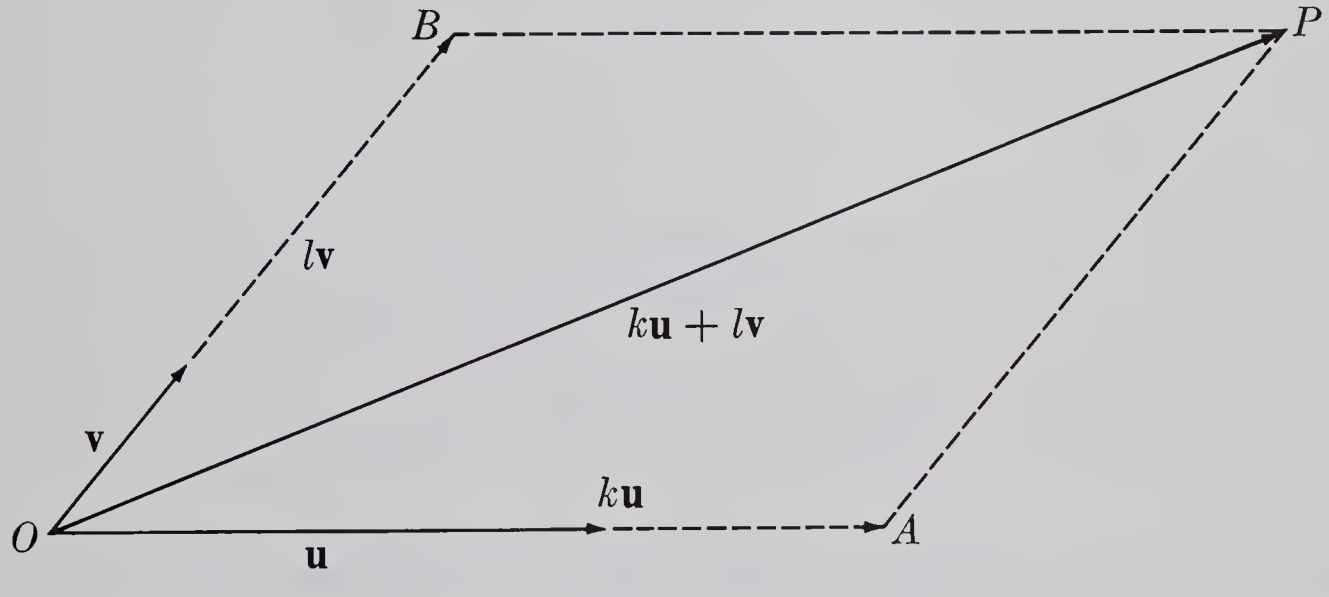


Figure 2.8

In the converse, given the point P in the plane determined by **u** and **v**, we need draw lines only through P parallel to **v** intersecting the line of **u** at A and through P parallel to **u** intersecting the line of **v** at B. Then $\overrightarrow{OA} = k\mathbf{u}$ for some k, and $\overrightarrow{OB} = l\mathbf{v}$ for some l.

Thus, for noncollinear vectors **u** and **v**, there is a one-to-one correspondence between the set

$$\{k\mathbf{u} + l\mathbf{v} \mid k, l \in Re\}$$

and the set of points in the plane through the origin determined by **u** and **v**. We call $k\mathbf{u} + l\mathbf{v}$ a linear combination of the vectors **u** and **v**.

Example 1. Sketch the plane through the origin determined by the vectors $\mathbf{u} = (2, 5, 4)$ and $\mathbf{v} = (-3, 1, 2)$. State the set of vectors determined by this plane.

Solution: The required plane is represented by the triangle OPQ in which $\overrightarrow{OP} = \mathbf{u}$ and $\overrightarrow{OQ} = \mathbf{v}$ (Figure 2.9). The set determined by this plane is

$$\{k(2, 5, 4) + l(-3, 1, 2) \mid k, l \in Re\} .$$

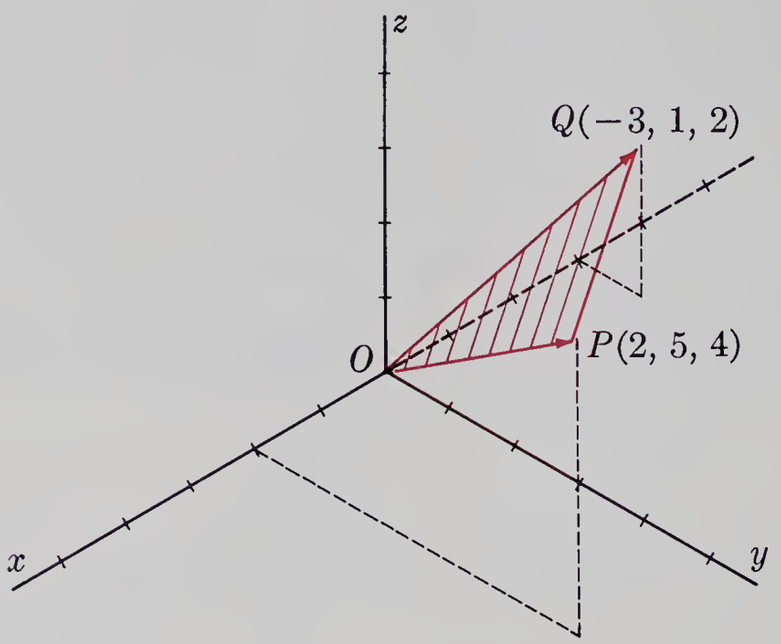


Figure 2.9

Three nonzero vectors are said to be coplanar if their geometric representatives with initial point at the origin lie in the same plane. The condition for three vectors to be coplanar is that at least one of the vectors be a linear combination of the other two.

Example 2. Show that the vector $(9, -10, -3)$ is a linear combination of the vectors $(-1, 2, 1)$ and $(3, -2, 0)$.

Solution: We must show that real numbers k and l exist such that

$$\begin{aligned}(9, -10, -3) &= k(-1, 2, 1) + l(3, -2, 0) \\ &= (-k, 2k, k) + (3l, -2l, 0) \\ &= (-k + 3l, 2k - 2l, k) .\end{aligned}$$

Thus we must find k and l such that

$$\begin{aligned}-k + 3l &= 9 , \\ 2k - 2l &= -10 , \\ k &= -3 .\end{aligned}$$

We find that $k = -3$ and $l = 2$, so that

$$(9, -10, -3) = -3(-1, 2, 1) + 2(3, -2, 0) .$$

The vectors $(9, -10, -3)$, $(-1, 2, 1)$, and $(3, -2, 0)$ are coplanar.

Example 3. Show that the vectors $(1, 0, 6)$, $(1, 2, -1)$, and $(3, 0, 4)$ are not coplanar; that is, none of the vectors is a linear combination of the others.

Solution: We can see at once that none of these vectors is a scalar multiple of either of the others. Thus any pair of the vectors, for example, the second and third, will determine a plane. Then, to show that $(1, 0, 6)$ is not in the plane they determine, we must show that $(1, 0, 6)$ is not a linear combination of $(1, 2, -1)$ and $(3, 0, 4)$; that is, we must show that it is impossible to find real numbers k and l such that

$$\begin{aligned}(1, 0, 6) &= k(1, 2, -1) + l(3, 0, 4) \\ &= (k, 2k, -k) + (3l, 0, 4l) \\ &= (k + 3l, 2k, -k + 4l) .\end{aligned}$$

Suppose it were possible; then the system

$$\begin{aligned}k + 3l &= 1 , \\ 2k &= 0 , \\ -k + 4l &= 6 ,\end{aligned}$$

would have a solution. From the second equation, $k = 0$ and so from the first, $l = \frac{1}{3}$. But this pair of values does not satisfy the third equation.

$$-0 + 4(\tfrac{1}{3}) = \tfrac{4}{3} \neq 6 .$$

The system of equations is inconsistent and has no solution; that is, k and l do not exist such that

$$(1, 0, 6) = k(1, 2, -1) + l(3, 0, 4) ,$$

and so $(1, 0, 6)$ is not coplanar with $(1, 2, -1)$ and $(3, 0, 4)$. Thus, $(1, 0, 6)$, $(1, 2, -1)$, and $(3, 0, 4)$ are noncoplanar.

EXERCISE 2.6

1. Sketch the planes through the origin determined by the following pairs of vectors and state the set of vectors determined by each plane.
   (a) $(1, 0, 0)$, $(0, 2, 0)$
   (b) $(1, 2, 0)$, $(2, 0, -1)$
   (c) $(3, 2, 2)$, $(-2, 5, -3)$
   (d) $(-3, 2, 4)$, $(1, 0, 3)$
   (e) $(-2, -2, 5)$, $(0, -3, 4)$
   (f) $(-4, 0, 0)$, $(3, -1, 5)$

2. Find k, if possible, so that
   (a) $2(-1, 3) - (k, 3) = (-7, 3)$,
   (b) $3(1, 1, -3) + 2(-3, k, 4) = (-3, 3, -1)$,
   (c) $3(-1, 1, 2) - 2(4, 1, k) = (-11, 0, 4)$.
   (d) $2(1, 2, -k) + (4, -1, 5) = 3(2, 1, k)$.

3. Establish whether the first vector in each of the following is coplanar with, that is, a linear combination of, the other two vectors.
   (a) $(-3, 1), (1, 0), (0, 1)$
   (b) $(2, 3), (0, -1), (1, 0)$
   (c) $(3, 5), (0, 1), (-1, 0)$
   (d) $(-3, 4), (0, 5), (2, 0)$
   (e) $(2, 3), (1, -1), (1, 0)$
   (f) $(-2, -2), (1, 2), (2, 1)$
   (g) $(2, 3, -5), (1, 0, 0), (0, 1, 0)$
   (h) $(2, 2, -5), (1, 1, 0), (0, 0, 1)$
   (i) $(3, 1, 1), (0, 1, 1), (1, 1, 0)$
   (j) $(3, 1, 1), (0, 1, 1), (1, 0, 0)$
   (k) $(3, 3, -1), (1, 1, 1), (1, 1, -1)$
   (l) $(10, -3, 1), (1, 3, -2), (4, 1, -1)$
   (m) $(10, -3, 0), (1, 3, -2), (4, 1, -1)$
   (n) $(-3, 11, -16), (1, 3, -4), (2, 1, -1)$

4. Are the following sets of vectors coplanar?
   (a) $(0, 0, 0), (3, -1, 2), (1, 1, 4)$
   (b) $(1, 1, -3), (4, -1, 2), (2, 2, -6)$
   (c) $(1, 0, 0), (0, 1, 0), (2, 3, 4)$
   (d) $(0, 1, -1), (5, 5, -5), (0, 5, -5)$

5. If three 3-dimensional vectors are coplanar, must each be a linear combination of the other two? (Refer to the results in question (4).)

6. Are the following sets of vectors coplanar?
   (a) $(3, -1, 1), (2, -1, 0), (3, 1, 4)$
   (b) $(-1, 0, 3), (5, 2, 1), (-1, 1, 2)$
   (c) $(0, -1, 3), (3, -1, 0), (3, 0, -1)$
   (d) $(1, 1, 1), (1, -1, 1), (3, 0, 4)$

2.7. Bases for V_2

In any set consisting of two vectors $\mathbf{u}$ and $\mathbf{v}$ in V_2, there are two possibilities: *either* $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other; that is, $\mathbf{u}$ and $\mathbf{v}$ are collinear, *or* $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other; that is, they are not collinear.

In the latter case, we have seen in Section 2.6 that the set of vectors

$$\{k\mathbf{u} + l\mathbf{v} \mid k, l \in Re\}$$

corresponds to the entire set of points in the plane determined by $\mathbf{u}$ and $\mathbf{v}$, that is, in this case, to the entire 2-space. This fact implies that any other vector in V_2 is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. It is this idea that we shall investigate in this section.

Example 1. Given the vectors $\mathbf{u} = (1, 2)$, $\mathbf{v} = (3, 5)$, and $\mathbf{w} = (2, 4)$, show that at least one of the vectors is a linear combination of the other two.

Solution: In this case, $\mathbf{u}$ and $\mathbf{w}$ are scalar multiples of each other and so it is obvious that $\mathbf{w} = 2\mathbf{u} + 0\mathbf{v}$.

Example 2. Given $\mathbf{u} = (1, 2)$, $\mathbf{v} = (3, 5)$, and $\mathbf{w} = (2, 6)$, show that at least one of the vectors is a linear combination of the others.

Solution: In this case, no two of the vectors are scalar multiples of each other. We must find k and l such that

$$\mathbf{u} = k\mathbf{v} + l\mathbf{w}\,,$$

that is,

$$\begin{aligned}(1, 2) &= k(3, 5) + l(2, 6)\\ &= (3k, 5k) + (2l, 6l)\\ &= (3k + 2l, 5k + 6l)\,.\end{aligned}$$

$$\begin{aligned}3k + 2l &= 1\,.\\ 5k + 6l &= 2\,.\end{aligned}$$

The solution of this system of equations is $k = \frac{1}{4}$, $l = \frac{1}{8}$. Therefore,

$$(1, 2) = \tfrac{1}{4}(3, 5) + \tfrac{1}{8}(2, 6)\ ;$$

that is,

$$\mathbf{u} = \tfrac{1}{4}\mathbf{v} + \tfrac{1}{8}\mathbf{w}\,.$$

Example 3. Show that, in Example 1, $\mathbf{v}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{w}$.

Solution: Geometrically, $\mathbf{u}$ and $\mathbf{w}$ are collinear vectors and any linear combination of $\mathbf{u}$ and $\mathbf{w}$ will be a vector collinear with $\mathbf{u}$ and $\mathbf{w}$. Since $\mathbf{v}$ is not collinear with $\mathbf{u}$ or $\mathbf{w}$, it follows that $\mathbf{v}$ cannot be a linear combination of $\mathbf{u}$ and $\mathbf{w}$. Algebraically, this means that it should be impossible to find k and l such that

$$\begin{aligned}(3, 5) &= k(1, 2) + l(2, 4)\\ &= (k, 2k) + (2l, 4l)\\ &= (k + 2l, 2k + 4l)\,,\end{aligned}$$

that is, such that

$$k + 2l = 3\,,$$

$$2k + 4l = 5\,.$$

It should be immediately obvious that this system of equations is inconsistent and does not have a solution.

Example 4. In Example 2, it was shown that $\mathbf{u} = \frac{1}{4}\mathbf{v} + \frac{1}{8}\mathbf{w}$. Show that this expression for $\mathbf{u}$ is unique; that is, that $\mathbf{u}$ is no other linear combination of $\mathbf{v}$ and $\mathbf{w}$.

Solution: It has already been pointed out that no two of the vectors are scalar multiples of each other. Now suppose that

$$\mathbf{u} = \tfrac{1}{4}\mathbf{v} + \tfrac{1}{8}\mathbf{w} \quad \text{and} \quad \mathbf{u} = a\mathbf{v} + b\mathbf{w},$$

where

$$(a, b) \neq (\tfrac{1}{4}, \tfrac{1}{8}).$$

Then

$$\tfrac{1}{4}\mathbf{v} + \tfrac{1}{8}\mathbf{w} = a\mathbf{v} + b\mathbf{w}$$

and

$$(\tfrac{1}{4} - a)\mathbf{v} = (b - \tfrac{1}{8})\mathbf{w}.$$

By assumption, either $a \neq \frac{1}{4}$ or $b \neq \frac{1}{8}$. Suppose $b \neq \frac{1}{8}$; then

$$\mathbf{w} = \frac{\frac{1}{4} - a}{b - \frac{1}{8}}\mathbf{v}.$$

This result violates the known fact that no two of the vectors are scalar multiples of each other. We conclude that no such real number pair (a, b) exists, and that

$$\mathbf{u} = \tfrac{1}{4}\mathbf{v} + \tfrac{1}{8}\mathbf{w}$$

is the only linear combination of $\mathbf{v}$ and $\mathbf{w}$ possible for $\mathbf{u}$.

Theorem. If $\mathbf{u}$ and $\mathbf{v}$ are noncollinear vectors of V_2, then any vector $\mathbf{w}$ in V_2 is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Proof: Obviously, $\mathbf{u}$ and $\mathbf{v}$ are both nonzero vectors. We must show that, if $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$ are not scalar multiplies of each other, and if $\mathbf{w} = (e, f)$ is any vector in V_2, then

$$\mathbf{w} = k\mathbf{u} + l\mathbf{v}$$

for some real numbers k and l. But

$$\begin{aligned}(e, f) &= k(a, b) + l(c, d)\\ &= (ka, kb) + (lc, ld)\\ &= (ka + lc, kb + ld)\end{aligned}$$

if and only if

$$\begin{aligned}ka + lc &= e,\\ kb + ld &= f.\end{aligned}$$

This system of equations will have a unique solution

$$k = \frac{cf - de}{bc - ad}, \qquad l = \frac{be - af}{bc - ad}$$

provided $bc - ad \neq 0$. But $bc - ad = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are collinear (see question (5), Exercise 2.7). Hence, k and l exist such that

$$\mathbf{w} = k\mathbf{u} + l\mathbf{v}\,.$$

Corollary. The expression for $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$ is unique.

A set of vectors such as those described in the above theorem is called a basis for V_2. Thus a basis for V_2 is any pair of noncollinear vectors of V_2; any vector of V_2 is a linear combination of the vectors of a basis.

The natural basis for V_2 is the pair of noncollinear vectors

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1)\,.$$

It is immediately obvious that

$$\begin{aligned}(a, b) &= a\,(1,0) + b\,(0, 1)\\ &= a\mathbf{e}_1 + b\mathbf{e}_2\,.\end{aligned}$$

EXERCISE 2.7

1. In the following sets, show that at least one vector is a linear combination of the other two.

   (a) $(0, 0), (3, 5), (2, -3)$ (b) $(0, 1), (1, 0), (3, -4)$

   (c) $(1, -1), (4, 7), (-3, 3)$ (d) $(4, 4), (-1, -1), (-3, -3)$

   (e) $(3, 7), (0, 0), (2, 9)$ (f) $(4, 2), (5, 9), (2, 1)$

   (g) $(1, 1), (3, 2), (1, 0)$ (h) $(3, -2), (0, 1), (5, 2)$

   (i) $(3, 2), (-1, 1), (5, 7)$ (j) $(-2, -1), (3, 1), (2, -5)$

2. Which of the following sets of vectors are bases for V_2? In the case of a basis, express $(5, -2)$ as a linear combination of the basis vectors.

   (a) $(0, 0), (3, 5)$ (b) $(1, 0), (0, 3)$ (c) $(-1, 5), (2, -10)$

   (d) $(-2, 0), (0, -3)$ (e) $(1, 3), (2, -6)$ (f) $(2, -4), (-1, 3)$

   (g) $(3, -1), (5, -2)$ (h) $(2, 0), (4, 2)$ (i) $(1, -1), (-1, 1)$

3. Express the following vectors as ordered pairs of real numbers.

   (a) $3\mathbf{e}_1 + 2\mathbf{e}_2$ (b) $-\mathbf{e}_1 + 4\mathbf{e}_2$ (c) $6\mathbf{e}_1$ (d) $-\mathbf{e}_1 - 2\mathbf{e}_2$

4. Show that a basis for V_2 cannot contain the zero vector.

5. Show that (a, b) and (c, d) are collinear if and only if $ad - bc = 0$. *Hint:* consider the following cases.

   (i) $a = b = c = d = 0$ (ii) $a = b = 0$

   (iii) $c = d = 0$ (iv) $a = c = 0$, b, d nonzero

   (v) $b = d = 0$, a, c nonzero (vi) a, b, c, d nonzero.

6. Prove the corollary to the theorem of this section as in Example 4.

2.8. Sets of Vectors in V_3

Let $\mathbf{u}$ and $\mathbf{v}$ be distinct nonzero vectors in V_3. There are two possibilities:

(i) $\mathbf{u}$ and $\mathbf{v}$ are collinear; then each vector is a scalar multiple of the other;

(ii) $\mathbf{u}$ and $\mathbf{v}$ are not collinear; they are not scalar multiples of each other.

In the latter case, $\mathbf{u}$ and $\mathbf{v}$ determine a plane through the origin in 3-space.

Example 1. $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{v} = (2, -6, -4)$ are collinear vectors;

$$\mathbf{u} = -\tfrac{1}{2}\mathbf{v} \quad \text{and} \quad \mathbf{v} = -2\mathbf{u}.$$

Example 2. $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{v} = (2, 0, -1)$ are not collinear; they determine a plane through the origin consisting of the vectors

$$\{a\mathbf{u} + b\mathbf{v} \mid a, b \in Re\}.$$

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are distinct nonzero vectors of V_3, there are three possibilities:

(i) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are collinear; then the vectors are scalar multiples of each other;

(ii) two of the vectors, say $\mathbf{u}$ and $\mathbf{v}$, determine a plane through the origin and the third is a vector in that plane; that is, $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$;

(iii) no one of the vectors is in the plane determined by the other two; that is, none of the vectors is a linear combination of the other two.

Example 3. $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (2, -6, -4)$, and $\mathbf{w} = (-5, 15, 10)$ are collinear vectors;

$$\mathbf{v} = -2\mathbf{u} \quad \text{and} \quad \mathbf{w} = 5\mathbf{u}.$$

Example 4. $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{v} = (2, 0, -1)$ determine a plane through the origin and

$$\begin{aligned}\mathbf{w} &= (-8, 6, 7) \\ &= 2(-1, 3, 2) + (-3)(2, 0, -1)\end{aligned}$$

is in this plane.

Example 5. $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 2, 0)$, and $\mathbf{w} = (0, 0, 3)$ are such that none of the vectors is in the plane determined by the other two; that is, none of the vectors is a linear combination of the other two, since a nonzero component cannot be a linear combination of two zero components.

Now consider a set of four distinct vectors of V_3. The possibilities are the following:

(i) all four vectors are collinear; that is, each vector is a scalar multiple of each of the others;

(ii) all four vectors are coplanar; that is, two of the vectors determine a plane and the other two are both in this plane;

(iii) three of the vectors are coplanar but the fourth does not lie in their plane;

(iv) no three of the vectors are coplanar.

The fourth possibility will be considered in the next section.

Example 6. The vectors **u, v, w,** as given in Example 3, and $\mathbf{x} = (-3, 9, 6)$ are four nonzero collinear vectors in V_3.

Example 7. The vectors **u, v, w,** given in Example 4, and

$$\begin{aligned}\mathbf{x} &= (5, -9, -7)\\ &= -3(-1, 3, 2) + (2, 0, -1)\\ &= -3\mathbf{u} + \mathbf{v}\end{aligned}$$

are such that **w** and **x** both lie in the plane determined by **u** and **v.**

Example 8. The vectors **u** and **v,** as given in Example 5, determine a plane and

$$\begin{aligned}\mathbf{x} &= (3, 4, 0)\\ &= 3\mathbf{u} + 2\mathbf{v}\end{aligned}$$

lies in this plane, but **w** does not; that is, we have three vectors that are coplanar and a fourth that does not lie in their plane.

EXERCISE 2.8

1. Determine whether the following vectors are collinear.

   (a) $(3, 1, 1), (-6, 2, 0)$ (b) $(4, -1, 3), (2, -\frac{1}{2}, -\frac{3}{2})$

   (c) $(4, -4, 2), (-2, 2, 1)$ (d) $(3, -1, 0), (-6, 2, 2)$

2. State whether the following vectors are collinear. If they are not, select two vectors which determine a plane. Then determine whether the third vector lies in that plane (if so, express the third vector as a linear combination of the other two).

   (a) $(1, -1, 0), (-2, 2, 0), (\frac{1}{2}, -\frac{1}{2}, 0)$ (b) $(1, -1, 0), (-3, 3, 0), (-1, 1, 1)$

   (c) $(1, 0, 0), (0, 1, 0), (1, 1, 0)$ (d) $(1, 0, 0), (0, 1, 0), (0, 1, 1)$

   (e) $(1, 0, 0), (0, 1, 0), (1, 0, 1)$ (f) $(0, 2, \frac{1}{2}), (0, 4, 1), (0, -16, -8)$

   (g) $(-3, 1, 2), (1, 1, 4), (-2, 1, 0)$ (h) $(1, -1, 4), (3, -1, 1), (1, -2, -1)$

3. State whether the following vectors are collinear, coplanar, or noncoplanar. If they are coplanar, express two of the vectors as linear combinations of the other two.
   (a) $(1, -1, 0)$, $(-3, 3, 0)$, $(4, -4, 0)$, $(\frac{1}{2}, -\frac{1}{2}, 0)$
   (b) $(-3, 1, 0)$, $(2, 1, 4)$, $(6, -2, 0)$, $(-12, 4, 0)$
   (c) $(2, 3, -1)$, $(5, 1, 2)$, $(-4, -6, 2)$, $(-10, -2, -4)$
   (d) $(1, 0, 3)$, $(0, 2, 0)$, $(-2, 0, -6)$, $(1, 2, 3)$
   (e) $(0, 0, -2)$, $(-3, 1, 0)$, $(0, 4, 0)$, $(1, 1, 1)$
   (f) $(-1, 2, 0)$, $(0, 3, -1)$, $(2, 0, -1)$, $(1, 2, 1)$
   (g) $(3, 1, 1)$, $(1, -1, 2)$, $(4, 0, 3)$, $(2, 2, -1)$

2.9. Bases for V_3

In Section 2.7, we showed that, in a set of three vectors of V_2, at least one of the vectors is a linear combination of the other two. Further, we showed that, if **u** and **v** are noncollinear vectors in V_2, then any vector **w** in V_2 is a *unique* linear combination of **u** and **v**.

In this section, we shall illustrate the following theorem (the proof in general follows the method of the example).

Theorem. If **u, v,** and **w** are three noncoplanar vectors of V_3, then any vector of V_3 is a linear combination of **u, v,** and **w.**

Corollary 1. In any set of four vectors of V_3, at least one of the vectors is a linear combination of the other three (see question (5), Exercise 2.9).

Example. Show that no three of the vectors

$$\mathbf{u} = (-1, 0, 3), \quad \mathbf{v} = (1, -1, 4), \quad \mathbf{w} = (3, -2, 1), \quad \mathbf{x} = (1, 1, -2)$$

are coplanar and that **x** is a linear combination of **u, v,** and **w.**

Solution: It can be seen immediately that none of the vectors is a scalar multiple of one of the others. Then **u** and **v** determine a plane, and we shall show that **w** does not lie in this plane. Assume that it does; that is, assume that real numbers k and l exist such that

$$\begin{aligned}(3, -2, 1) &= k(-1, 0, 3) + l(1, -1, 4)\\ &= (-k, 0, 3k) + (l, -l, 4l)\\ &= (-k + l, -l, 3k + 4l)\,,\end{aligned}$$

and hence

$$\begin{aligned}-k + l &= 3\,,\\ -l &= -2\,,\\ 3k + 4l &= 1\,.\end{aligned}$$

The first two of these equations have the solution $k = -1$, $l = 2$, and substitution of these values in the third equation shows that this system of three equations in k and l is inconsistent and does not have a solution. Thus, **w** does not lie in the plane determined by **u** and **v**.

We must now show that **x** is not coplanar with any pair of the vectors **u, v,** and **w**. This involves showing, possibly as above, that {**u, v, x**}, {**u, w, x**}, and {**v, w, x**} are not sets of coplanar vectors.

However, rather than carry out the above analysis three times, we proceed as follows. We express **x** as a linear combination, $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, of **u, v,** and **w**.

Suppose that one of a, b, or c is zero. Then we may write **x** as a linear combination of two of the vectors. Therefore, **x** is coplanar with these vectors.

Suppose that none of a, b, and c is zero. We shall argue that **x** cannot be coplanar with any pair of the vectors **u, v,** and **w**.

We wish to find real numbers a, b, and c such that

$$\begin{aligned}(1,1,-2) &= a(-1,0,3) + b(1,-1,4) + c(3,-2,1)\\ &= (-a,0,3a) + (b,-b,4b) + (3c,-2c,c)\\ &= (-a+b+3c, -b-2c, 3a+4b+c)\,,\end{aligned}$$

that is, such that

$$\begin{aligned}-a + b + 3c &= 1\,,\\ -b - 2c &= 1\,,\\ 3a + 4b + c &= -2\,.\end{aligned}$$

We now have the problem of finding the solution of a system of three equations in the variables a, b, and c. By the method of elimination or any suitable method, the solution is found to be $a = -4$, $b = 3$, $c = -2$, so that

$$(1,1,-2) = -4(-1,0,3) + 3(1,-1,4) - 2(3,-2,1)\,.$$

We have now expressed **x** as a linear combination of **u, v,** and **w,** and we shall now show that this expression is unique. Assume that it is not; that is, assume that

$$-4(-1,0,3) + 3(1,-1,4) - 2(3,-2,1) = d(-1,0,3) + e(1,-1,4) + f(3,-2,1),$$

where $(-4,3,-2) \neq (d,e,f)$. Then

$$[d+4](-1,0,3) + [e-3](1,-1,4) + [f+2](3,-2,1) = (0,0,0)\,.$$

Now either $d \neq -4$ or $e \neq 3$ or $f \neq -2$. If $d \neq -4$, we have

$$\begin{aligned}(-1,0,3) &= -\frac{e-3}{d+4}(1,-1,4) - \frac{f+2}{d+4}(3,-2,1)\\ &= -\frac{e-3}{d+4}\mathbf{v} - \frac{f+2}{d+4}\mathbf{w}.\end{aligned}$$

This result gives us a contradiction since it has been shown that $\mathbf{u}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{w}$. (Similar contradictions exist if $e \neq 3$ or $f \neq -2$.) This contradiction means that it is impossible to have

$$\mathbf{x} = d\mathbf{u} + e\mathbf{v} + f\mathbf{z}$$

in which one of d, e, and f is 0 and so $\mathbf{x}$ is not coplanar with any pair of vectors of the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

In the study of this example, we have seen verified the following corollary to the theorem of this section.

Corollary 2. The expression of any vector in V_3 as a linear combination of three noncoplanar vectors of V_3 is unique.

A set of three noncoplanar vectors of V_3 is called a basis for V_3; any vector in V_3 can be expressed uniquely as a linear combination of a set of basis vectors.

The natural basis for V_3 is the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1) .$$

It is immediately seen that these vectors are noncoplanar and that any vector $\mathbf{u} = (a, b, c)$ of V_3 can be expressed as

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) ;$$

that is,

$$\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 .$$

There should be no confusion in our using $\mathbf{e}_1$, $\mathbf{e}_2$ as the symbols for the natural basis for V_2 and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ as symbols for the natural basis for V_3. In each case it should be known whether we are discussing two- or three-dimensional vectors.

EXERCISE 2.9

1. Do the following sets of vectors form bases for V_3; that is, are they noncoplanar?
   (a) (0, 1, 0), (1, 0, 0), (0, 0, 2)
   (b) (−1, 0, 0), (0, 3, 0), (0, 0, −2)
   (c) (−1, 0, 1), (0, 1, 0), (0, 0, 1)
   (d) (−1, 0, 1), (0, 0, 1), (1, 0, 0)
   (e) (1, −1, 2), (3, −1, 5), (−2, 2, −4)
   (f) (3, −1, 1), (2, −1, 1), (1, 1, 2)
2. Express the vector (1, 1, 1) as a linear combination of the basis vectors in question (1).
3. Express the following vectors as triples of real numbers.
   (a) $3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$
   (b) $-4\mathbf{e}_1 + 2\mathbf{e}_3$
   (c) $5\mathbf{e}_1 - 3\mathbf{e}_2 - \mathbf{e}_3$

4. Express the fourth vector in each of the following sets as a linear combination of the first three if possible.
   (a) $(1, 0, 0), (0, 2, 0), (0, 0, 3), (-1, 2, 1)$
   (b) $(3, 1, 0), (0, 3, 1), (1, 0, 3), (2, 2, 1)$
   (c) $(-3, 2, 1), (-3, 0, 0), (0, 2, 1), (1, 1, -2)$
   (d) $(4, -1, 1), (3, 0, -1), (0, 1, -1), (1, 1, 4)$
   (e) $(3, -3, 1), (2, 0, 0), (1, 3, 0), (5, 2, 1)$

5. Prove Corollary 1 of the theorem in this section.

6. Prove Corollary 2 of the theorem.

Chapter Summary

Cartesian co-ordinate system in three dimensions · Two- and three-dimensional algebraic vectors: V_2 and V_3 · Equality of vectors · Addition of vectors; multiplication of a vector by a scalar; geometric interpretation · Identification of algebraic and geometric vectors · Lines through the origin; collinear vectors · Planes through the origin; coplanar vectors · Bases for V_2 and V_3

REVIEW EXERCISE 2

1. Express the following as single vectors.
   (a) $4(3, -1) + 3(-5, 2)$
   (b) $3(1, 0, -5) + 2(6, -3, -3)$
   (c) $5(2, 2) - 4(3, -1) + \frac{1}{2}(4, 8)$
   (d) $-2(1, 1, 5) - 3(2, -1, 4) + 2(0, 3, 5)$
   (e) $3\mathbf{e}_1 - 5\mathbf{e}_2 + 6\mathbf{e}_3$
   (f) $6\mathbf{e}_1 - 4\mathbf{e}_2 - 5\mathbf{e}_3$

2. What algebraic vectors correspond to geometric vectors $\overrightarrow{PQ}$ if P and Q are the following points?
   (a) $P(6, -1), Q(-1, 6)$
   (b) $P(-2, -4), Q(3, -6)$
   (c) $P(-3, 1, -3), Q(4, -1, 4)$
   (d) $P(-1, -1, -5), Q(3, -2, 6)$

3. Find the following vectors geometrically.
   (a) $3(1, 1) + 2(-1, 3)$
   (b) $-2(3, 1) + 4(2, 2)$

4. Find the values of the variable involved (if possible) so that the following pairs of vectors are collinear.
   (a) $(6, -3), (4, k)$
   (b) $(3, -2, -4), (\frac{1}{2}, k, l)$

5. Establish whether the first vector in each of the following is a linear combination of the other two vectors.
   (a) $(3, 5), (-2, 4), (1, 1)$ (b) $(4, 1), (3, -1), (-6, 2)$
   (c) $(1, 3, -3), (4, 2, 1), (1, 0, 1)$ (d) $(2, 3, -4), (6, -1, 3), (-4, 1, 1)$

6. Are the following sets of vectors coplanar?
   (a) $(3, 2, -1), (-6, -4, 2), (1, 1, 1)$ (b) $(1, 0, 0), (0, 5, 0), (1, 1, 5)$
   (c) $(3, -1, 0), (1, 0, 1), (0, -1, 3)$ (d) $(1, 1, 2), (3, -1, 1), (4, 2, 2)$

7. Which of the following sets of vectors are bases for V_2? In the case of a basis, express $(4, -3)$ as a linear combination of the basis vectors.
   (a) $(0, 0), (2, 1)$ (b) $(-1, 3), (2, 1)$
   (c) $(-4, 2), (2, -1)$ (d) $(3, -1), (3, -2)$
   (e) $(2, 0), (0, 5)$ (f) $(1, 1), (2, 3)$

8. State whether the following vectors are collinear. If they are not, select two vectors which determine a plane. Then determine whether or not the third vector lies in that plane.
   (a) $(1, 1, -2), (2, 2, -4), (3, 3, 6)$ (b) $(1, 3, 1), (-2, -6, -2), (3, 9, 3)$
   (c) $(1, 1, 0), (0, 1, 1), (1, 0, 1)$ (d) $(4, 1, 1), (3, -1, 1), (2, -1, 1)$

9. State whether the following vectors are collinear, coplanar, or noncoplanar.
   (a) $(1, 0, 0), (3, 0, 0), (0, 2, 0), (0, 4, 0)$
   (b) $(1, 0, 0), (-3, 0, 0), (0, 2, 0), (0, 0, 4)$
   (c) $(2, -1, 1), (4, -2, 2), (1, 1, 0), (2, 1, 3)$

10. Do the first three vectors of each of the following sets form a basis for V_3? If so, express the fourth vector in each set as a linear combination of the first three.
   (a) $(0, 2, 0), (3, 0, 0), (0, 0, -2), (5, 1, -2)$
   (b) $(3, -1, 0), (-1, 0, 3), (0, -1, 3), (2, 2, -5)$
   (c) $(4, -1, 2), (1, 0, 3), (2, -1, 0), (3, 2, 3)$

Chapter 3

LENGTH AND INNER PRODUCT

3.1. Length of Line Segments in One and Two Dimensions

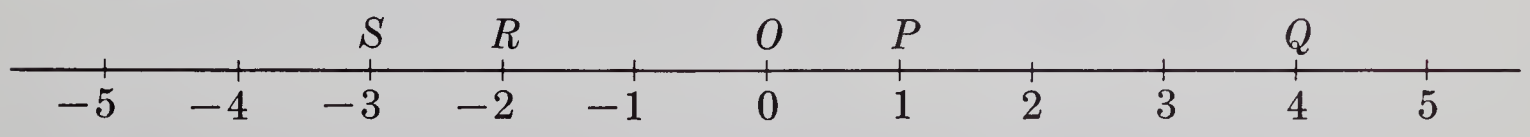
In this chapter, we shall assume that a fixed unit of measurement has been established. We can then set up a frame of reference to provide co-ordinates for points in one-, two-, and three-dimensional space. In one-dimensional space, that is, along a line, this unit of measurement allows us to assign the integer 1 to a certain point of the line after an origin for measurement has been established on the line, and hence to set up the real number line. With this fixed unit of measurement, we then make the following definition.

DEFINITION. The distance $d(P, Q)$ between the two points P and Q on the number line representing the real numbers x_1 and x_2, is defined to be

$$d(P, Q) = |x_2 - x_1|.$$

This definition assigns to each pair of points on the number line a unique nonnegative real number called the distance between the two points. This real number is positive if and only if the two points are distinct.

Example 1.



$d(O, P) = |1 - 0| = 1.$ $\quad$ $d(P, O) = |0 - 1| = 1.$

$d(O, Q) = |4 - 0| = 4.$ $\quad$ $d(Q, O) = |0 - 4| = 4.$

$d(P, Q) = |4 - 1| = 3.$ $\quad$ $d(Q, P) - |1 - 4| = 3.$

$d(P, R) = |-2 - 1| = 3.$ $\quad$ $d(R, P) = |1 - (-2)| = 3.$

$d(Q, S) = |-3 - 4| = 7.$ $\quad$ $d(S, Q) = |4 - (-3)| = 7.$

$d(Q, Q) = |4 - 4| = 0.$ $\quad$ $d(S, S) = |-3 - (-3)| = 0.$

$$d(O, P) + d(P, Q) = 1 + 3 = 4 = d(O, Q).$$

$$d(S, P) + d(P, Q) = 4 + 3 = 7 = d(S, Q).$$

$$d(P, R) + d(R, Q) = 3 + 6 = 9 > 3 = d(P, Q).$$

An examination of the results of Example 1 verifies that the distance $d(P, Q)$ we have defined for ordered pairs of points (P, Q) satisfies the following properties.

(i) $d(P, Q) > 0$ if $P \neq Q$.

(ii) $d(P, Q) = 0$ if $P = Q$.

(iii) $d(P, Q) = d(Q, P)$.

(iv) $d(P, Q) + d(Q, R) \geq d(P, R)$.

DEFINITION. The length PQ of a line segment with endpoints P and Q is defined to be the distance between the endpoints.

$$PQ = d(P, Q).$$

Example 2. Using the points O, P, Q, R, S of Example 1,

$OP = PO = 1$,
$PQ = QP = 3$,
$QS = SQ = 7$,
$OP + PQ = OQ$,
$PR + RQ > PQ$.

$OQ = QO = 4$,
$PR = RP = 3$,
$QQ = SS = 0$,
$SP + PQ = SQ$,

Note that these results for length agree with those we have used for many years based on a less formal definition of distance.

It follows immediately, from the properties of the distance $d(P, Q)$, that

(i) $PQ > 0$ if $P \neq Q$,

(ii) $PQ = 0$ if $P = Q$,

(iii) $PQ = QP$,

(iv) $PQ + QR \geq PR$.

We now use the definition of the length of a line segment in one dimension and the theorem of Pythagoras, to determine the length of a line segment, and hence the distance between two points, in two dimensions, that is, in the plane.

Theorem. The distance $d(P, Q)$ between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane and the length of the line segment PQ is

$$d(P, Q) = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane (Figure 3.1). The relative positions of the particular points shown in the diagram are not important; the points may be in any position.

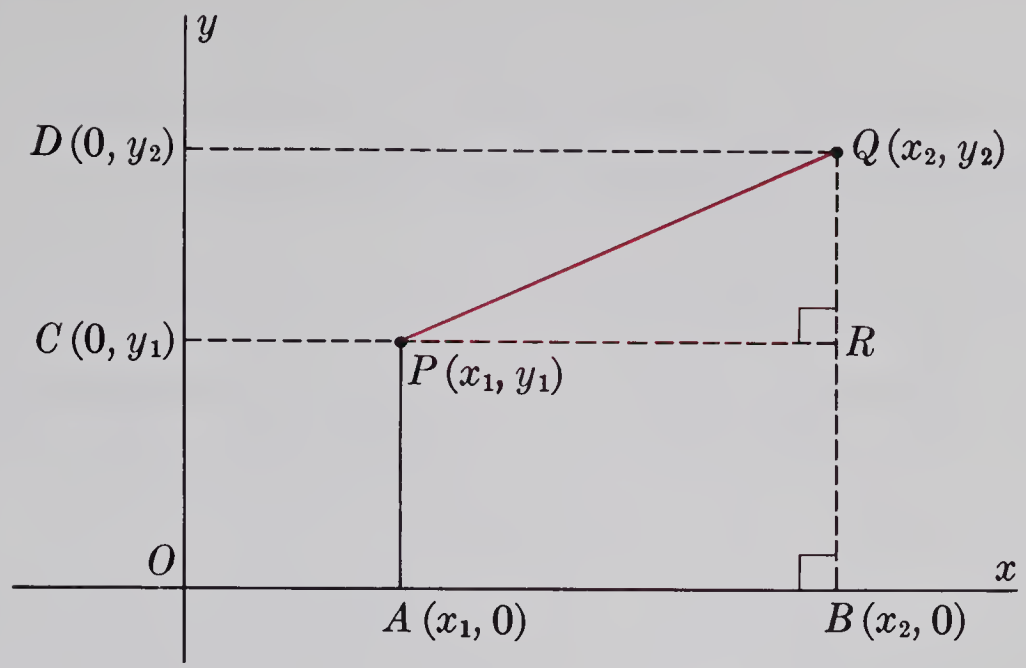


Figure 3.1

Construction: Draw PA and $QB \perp x$-axis, and PC and $QD \perp y$-axis; extend CP and QB to intersect in R. Then

$$PR = AB = |x_2 - x_1|$$

and

$$RQ = CD = |y_2 - y_1|.$$

Proof: By the theorem of Pythagoras,

$$PQ = \sqrt{PR^2 + RQ^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2},$$

or simply

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

It follows by our definition that $d(P, Q)$ has this same value.

Example 3. Let $P(2, 3)$, $Q(-1, 4)$, and $R(-3, -5)$ be three points in the plane.

(a) Find PQ, QR, and PR.
(b) Check that $PQ + QR \geq PR$.

Solution:

(a) $$PQ = \sqrt{(-1 - 2)^2 + (4 - 3)^2} = \sqrt{10}.$$

(b) $$QR = \sqrt{(-3 + 1)^2 + (-5 - 4)^2} = \sqrt{85}.$$
$$PR = \sqrt{(-3 - 2)^2 + (-5 - 3)^2} = \sqrt{89}.$$
$$PQ \simeq 3.162, \quad QR \simeq 9.220, \quad PR \simeq 9.434.$$

Therefore,

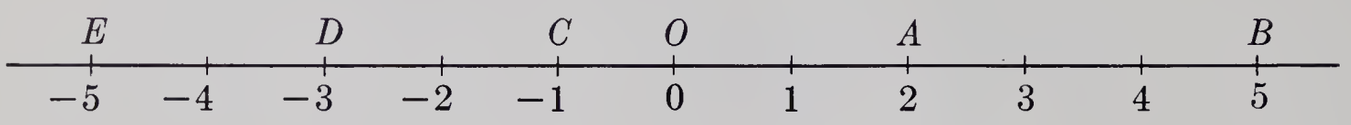
$$PQ + QR > PR.$$

EXERCISE 3.1

1. Find

   $d(A, B)$, OC, $d(D, C)$, AD, $d(B, B)$, AC, $d(E, D)$, $d(C, D)$, DA, CB

   for the points and line segments of the following diagram.



2. Verify that

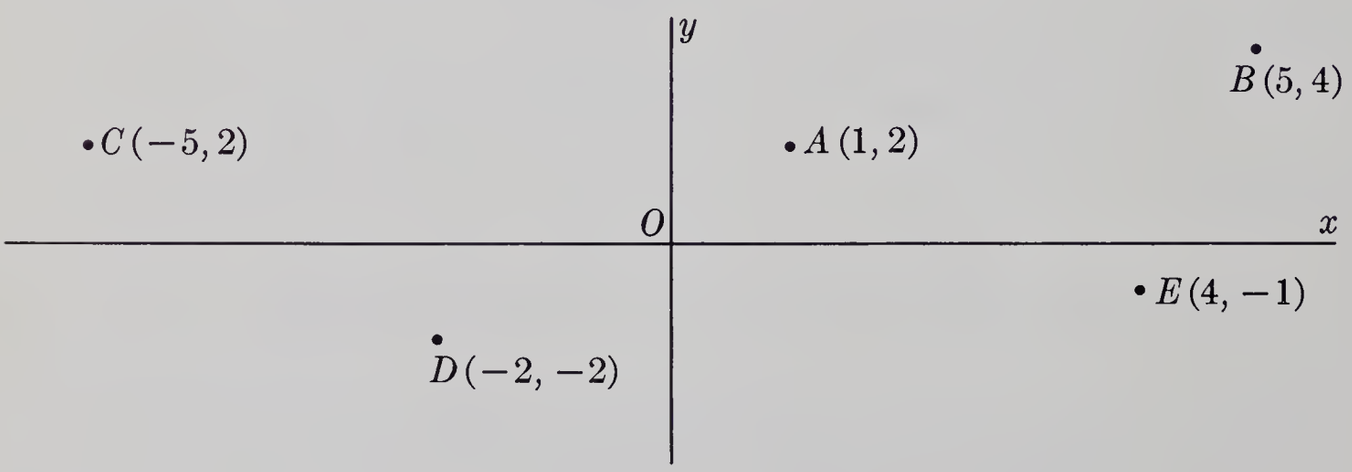
   (a) $ED + DA \geq EA$
   (b) $d(O, C) + d(C, B) \geq d(O, B)$
   (c) $AD + DB \geq AB$
   (d) $d(C, D) + d(D, A) \geq d(C,A)$
   (e) $CE + ED \geq CD$
   (f) $d(E, D) + d(C, D) \geq d(E, C)$
   (g) $DC + CA \geq DA$
   (h) $d(A, E) + d(E, D) \geq d(A, D)$

   for the points and line segments in the above diagram.

3. Find

   $d(A, C)$, AB, $d(D, E)$, BC, $d(E, A)$, $d(A, D)$, CE, $d(E, E)$

   for the points and line segments of the following diagram.



4. Verify the following for the points given in question (3).

   (a) $AB + BE \geq AE$
   (b) $d(C, A) + d(A, B) \geq d(C, B)$
   (c) $DA + AB \geq DB$
   (d) $d(E, A) + d(A, C) \geq d(E, C)$

5. Prove the theorem of this section using a diagram in which P is in quadrant IV and Q in quadrant II.

6. Prove that, for points P, Q, and R on a number line,

   $$d(P, Q) + d(Q, R) \geq d(P, R) .$$

   (Consider all possible cases.)

7. For points P and Q in the plane, prove that
   (a) $PQ > 0$ if $P \neq Q$,
   (b) $PQ = 0$ if and only if $P = Q$,
   (c) $PQ = QP$.

8. For what points P, Q, and R in the plane will $PQ + QR = PR$?

9. Use the result of question (8) to determine which of the following sets of points are collinear.
   (a) $P(3, -1)$, $Q(2, 5)$, $R(8, 3)$
   (b) $P(-1, 1)$, $Q(5, -1)$, $R(4, 3)$
   (c) $P(-1, 2)$, $Q(4, -2)$, $R(-3, 0)$

10. Determine which of the triangles PQR are right angled.
   (a) $P(1, 1)$, $Q(1, 4)$, $R(5, 4)$
   (b) $P(2, 3)$, $Q(-1, 2)$, $R(-3, -5)$
   (c) $P(-1, 2)$, $Q(5, 2)$, $R(1, -2)$

3.2. Line Segments in Three-Space

Theorem. The distance $d(P, Q)$ between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space and the length of the line segment PQ is

$$d(P, Q) = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof: Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be arbitrary points in 3-space. In proceeding from point P to point Q, we move $x_2 - x_1$ units in the x direction, $y_2 - y_1$ units in the y direction, and $z_2 - z_1$ units in the z direction (these are shown as positive quantities in Figure 3.2 but of course any or all of them could be negative).

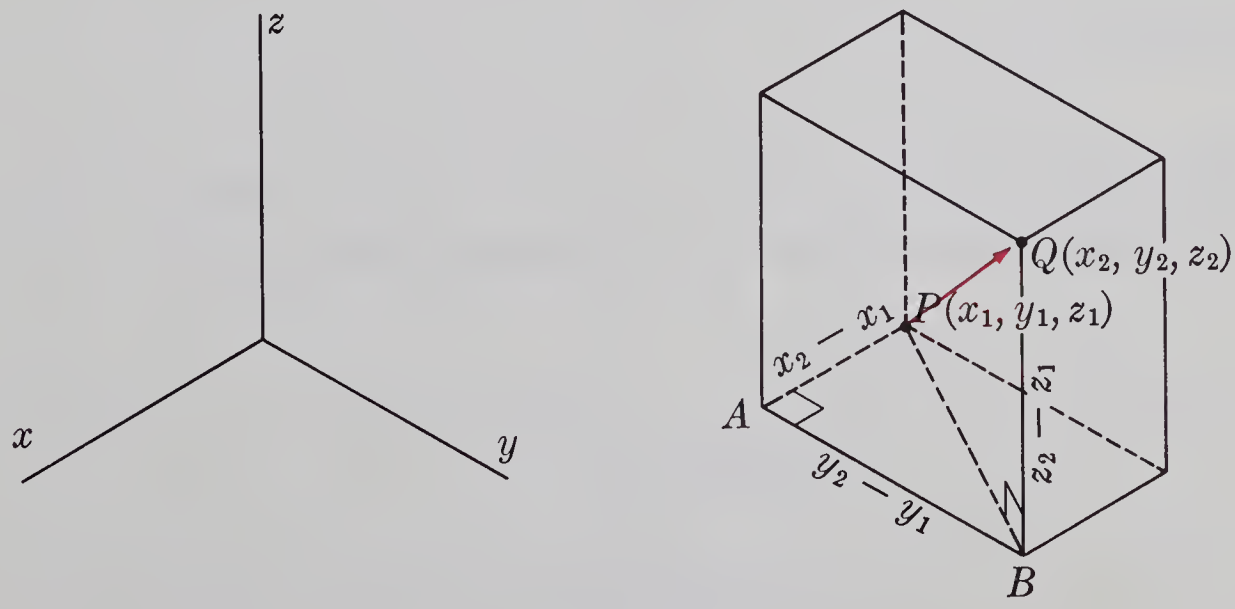


Figure 3.2

Line segment PA is parallel to the x-axis and has length $|x_2 - x_1|$; line segment AB is parallel to the y-axis and has length $|y_2 - y_1|$. Further, line segments PA and AB determine a plane parallel to the xy-plane and PB lies in this plane. From Section 3.1, line segment PB has length

$$PB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,.$$

Now line segment PQ is parallel to the z-axis and has length $|z_2 - z_1|$. This line segment is perpendicular to PB and so PB and BQ determine a plane which contains the right-angled triangle PBQ. Applying the theorem of Pythagoras, we obtain

$$\begin{aligned} PQ &= \sqrt{PB^2 + BQ^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= d(P, Q)\,. \end{aligned}$$

This gives us a formula for the distance between two points P and Q in space, that is, the length of the line segment PQ.

Example. Let $P(2, 3, -1)$, $Q(-1, 0, 5,)$, and $R(-2, -1, 4)$ be three points in 3-space.

(a) Find PQ, QR, and PR.

(b) Verify that $PQ + QR \geq PR$.

Solution:

(a) $PQ = \sqrt{(-1-2)^2 + (0-3)^2 + (5+1)^2} = \sqrt{54} = 3\sqrt{6}$.

$QR = \sqrt{(-2+1)^2 + (-1-0)^2 + (4-5)^2} = \sqrt{3}$.

$PR = \sqrt{(-2-2)^2 + (-1-3)^2 + (4+1)^2} = \sqrt{57}$.

(b) $PQ \simeq 7.348$, $QR \simeq 1.732$, $PR \simeq 7.550$.

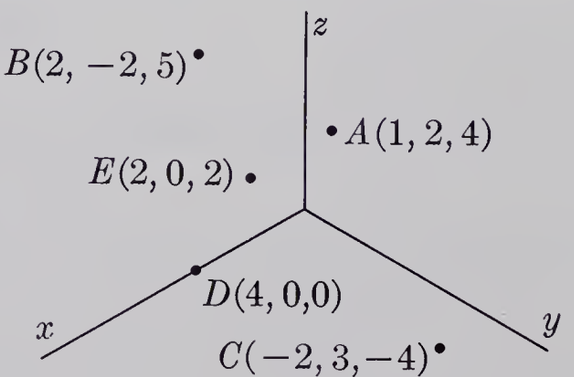
Therefore $PQ + QR > PR$.

EXERCISE 3.2

1. Find

   $d(A, B)$, OC, $d(D, C)$, AD, $d(B, B)$, AC, $d(E, D)$, $d(C, D)$, DA, CB

   for the points and line segments of the following diagram.



2. Verify that
   (a) $ED + DA \geq EA$ (b) $d(O, C) + d(C, B) \geq d(O, B)$
   (c) $AD + DB \geq AB$ (d) $d(C, D) + d(D, A) \geq d(C,A)$
   (e) $CE + ED \geq CD$ (f) $d(E, D) + d(D, C) \geq d(E, C)$
   (g) $DC + CA \geq DA$ (h) $d(A, E) + d(E, D) \geq d(A, D)$
   for the points and line segments in the diagram of question (1).

3. For points P and Q in 3-space, prove that
   (a) $PQ > 0$ if $P \neq Q$,
   (b) $PQ = 0$ if and only if $P = Q$,
   (c) $PQ = QP$.

4. For what points P, Q, and R in 3-space will $PQ + QR = PR$?

5. Use the result of question (4) to determine which of the following sets of points are collinear.
   (a) $P(-3, 1, 1)$, $Q(2, -3, 0)$, $R(-4, -1, 2)$
   (b) $P(-2, 1, 3)$, $Q(3, -1, 1)$, $R(10, -4, -4)$
   (c) $P(3, 1, -1)$, $Q(4, 0, -2)$, $R(3, -1, 0)$

6. Determine which of the following triangles PQR are right-angled.
   (a) $P(3, -1, 2)$, $Q(4, 3, -1)$, $R(3, 1, -4)$
   (b) $P(-1, -1, 3)$, $Q(4, -2, 5)$, $R(1, -3, -3)$
   (c) $P(3, -1, 0)$, $Q(4, 4, 1)$, $R(0, -2, 1)$

3.3. The Length of a Vector

In Section 2.3 it was pointed out that the two dimensional algebraic vector (a, b) could be represented geometrically in the plane by the directed line segment or arrow with initial point at the origin and terminal point at the point $P(a, b)$. Similarly, the three-dimensional vector (a, b, c) can be represented geometrically in 3-space by the line segment from the origin to the point $P(a, b, c)$.

Now the line segment from the origin to the point $P(a, b)$ has length $\sqrt{a^2 + b^2}$, and the line segment from the origin to the point $P(a, b, c)$ has length $\sqrt{a^2 + b^2 + c^2}$. We use these results to guide us in defining a length for the algebraic vectors (a, b) and (a, b, c).

DEFINITION. If $\mathbf{v} = (a, b)$, the length or magnitude $|\mathbf{v}|$ of $\mathbf{v}$ is defined to be

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$

If $\mathbf{v} = (a, b, c)$, the length $|\mathbf{v}|$ of $\mathbf{v}$ is defined to be

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}.$$

Example 1. Find the length of the vectors

(a) $\mathbf{u} = (3, -7)$,

(b) $\mathbf{v} = (-2, -3, -1)$.

Solution:

(a) $|\mathbf{u}| = \sqrt{3^2 + (-7)^2} = \sqrt{58}$.

(b) $|\mathbf{v}| = \sqrt{(-2)^2 + (-3)^2 + (-1)^2} = \sqrt{14}$.

Example 2.

(a) Find the length of the vector represented by the line segment from $P(-1, -1, 4)$ to $Q(3, -7, 1)$.

(b) Find the distance between the points $P(-1, -1, 4)$ and $Q(3, -7, 1)$.

Solution:

(a) Let

$$\mathbf{v} = \overrightarrow{PQ};$$

then

$$\mathbf{v} = (3 + 1, \ -7 + 1, \ 1 - 4) = (4, -6, -3)$$

and

$$|\mathbf{v}| = \sqrt{4^2 + (-6)^2 + (-3)^2} = \sqrt{61}.$$

(b)
$$d(P, Q) = \sqrt{(3 + 1)^2 + (-7 + 1)^2 + (1 - 4)^2}$$

$$= \sqrt{4^2 + (-6)^2 + (-3)^2} = \sqrt{61}.$$

Example 2 merely emphasizes the relation between the defined length or magnitude of an algebraic two- or three-dimensional vector and the length of the corresponding line segments in 2-space or 3-space.

Example 3. Given $\mathbf{u} = 3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{v} = -\mathbf{e}_1 + \mathbf{e}_3$ in V_3, find

(a) $|\mathbf{u}|$, (b) $|\mathbf{v}|$, (c) $|\mathbf{u} + \mathbf{v}|$, (d) $|2\mathbf{u} - 3\mathbf{v}|$.

Solution:

(a) $\mathbf{u} = 3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3 = (3, -2, 1)$.

$$|\mathbf{u}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}.$$

(b) $\mathbf{v} = -\mathbf{e}_1 + \mathbf{e}_3 = (-1, 0, 1)$.

$$|\mathbf{v}| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}.$$

(c) $\mathbf{u} + \mathbf{v} = (3, -2, 1) + (-1, 0, 1) = (2, -2, 2)$.

$$|\mathbf{u} + \mathbf{v}| = \sqrt{2^2 + (-2)^2 + 2^2} = \sqrt{12} = 2\sqrt{3}.$$

(d) $2\mathbf{u} - 3\mathbf{v} = 2(3, -2, 1) - 3(-1, 0, 1)$
$= (6, -4, 2) + (3, 0, -3)$
$= (9, -4, -1).$

$$|2\mathbf{u} - 3\mathbf{v}| = \sqrt{9^2 + (-4)^2 + (-1)^2} = \sqrt{98} = 7\sqrt{2}.$$

EXERCISE 3.3

1. Find the lengths of the following vectors.
   (a) $(0, 2)$ (b) $(0, 3, 0)$ (c) $(-6, 0)$ (d) $(0, 0, -4)$
   (e) $(1, 5)$ (f) $(1, 1, 2)$ (g) $(-3, 4)$ (h) $(-3, -3, 3)$
   (i) $(-2, -6)$ (j) $(4, -1, 6)$ (k) $(-5, -4)$ (l) $(\sqrt{2}, \sqrt{3}, \sqrt{4})$
   (m) $4\mathbf{e}_1 - 2\mathbf{e}_2$ (n) $-\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$
   (o) $2\mathbf{e}_1 - 3\mathbf{e}_2$ (p) $2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$
2. Find the length of the vector $\overrightarrow{PQ}$ for each of the following P and Q.
   (a) $P(0, 0)$, $Q(-2, 3)$ (b) $P(-1, 5)$, $Q(3, -2)$
   (c) $P(-2, -3)$, $Q(-5, -1)$ (d) $P(0, 0, 0)$, $Q(-3, 1, 4)$
   (e) $P(-1, 2, 2)$, $Q(3, 1, -4)$ (f) $P(-3, 1, -3)$, $Q(4, 1, -7)$
3. If $\mathbf{u} = (3, 1, -4)$, $\mathbf{v} = 2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, and $\mathbf{w} = \mathbf{e}_1 + 4\mathbf{e}_2 - 2\mathbf{e}_3$, find the following.
   (a) $|\mathbf{u}|$ (b) $|\mathbf{v}|$ (c) $|\mathbf{w}|$ (d) $|\mathbf{u} + \mathbf{v}|$ (e) $|\mathbf{v} + \mathbf{w}|$
   (f) $|\mathbf{u} + \mathbf{w}|$ (g) $|\mathbf{u} + \mathbf{v} + \mathbf{w}|$ (h) $|2\mathbf{u} - 3\mathbf{v}|$ (i) $|3\mathbf{u} - \mathbf{v} - 2\mathbf{w}|$
4. Prove that $|k\mathbf{v}| = |k||\mathbf{v}|$ if k is a real number. Explain the difference in the significance of the bars in the symbols $|k|$ and $|\mathbf{v}|$.
5. Prove that the length of a vector can equal 0 if and only if the vector is the zero vector.

3.4. Unit Vectors

The vector $\mathbf{u} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ has length

$$|\mathbf{u}| = \sqrt{\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3}} = 1.$$

Such a vector is called a unit vector.

DEFINITION. A unit vector is a vector of length 1.

Example 1.
(a) $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors.
(b) $\mathbf{u} = (\frac{3}{5}, -\frac{4}{5})$ is a unit vector since $|\mathbf{u}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$.
(c) $\mathbf{v} = (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$ is a unit vector since $|\mathbf{v}| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$.

It should be obvious that every line segment in the plane joining the origin to a point on the circle whose equation is $x^2 + y^2 = 1$ will be a unit vector; that is, there is a unit vector in every direction in the plane (Figure 3.3).

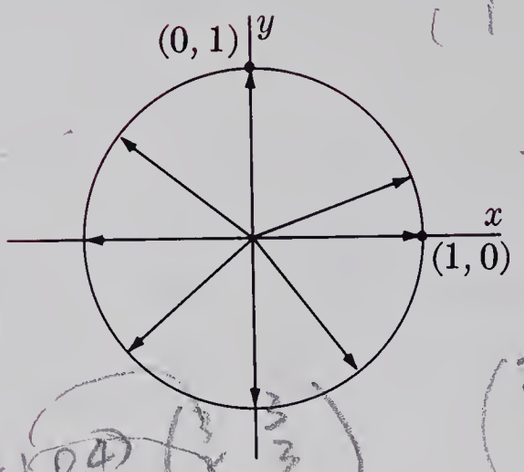


Figure 3.3

This fact means that we should be able to find a unit vector in the direction of an arbitrary vector (a, b).

This unit vector is found by using the result of question (4), Exercise 3.3.

$$|k\mathbf{v}| = |k|\,|\mathbf{v}|\,;$$

that is, the length of a scalar multiple of a vector is the length of the vector multiplied by the absolute value of the scalar. For example,

$$|3\mathbf{v}| = 3\,|\mathbf{v}|\,; \quad |-5\mathbf{u}| = |-5|\,|\mathbf{u}| = 5\,|\mathbf{u}|\,.$$

Now consider the vector $\mathbf{u} = (4, -3)$. Its length is

$$|\mathbf{u}| = \sqrt{4^2 + (-3)^2} = 5,$$

so that $\frac{1}{5}\mathbf{u}$ and $-\frac{1}{5}\mathbf{u}$ will have length $\frac{1}{5}$ that of $\mathbf{u}$, that is, 1.

$$|\tfrac{1}{5}\mathbf{u}| = |(\tfrac{4}{5}, -\tfrac{3}{5})| = \sqrt{\tfrac{16}{25} + \tfrac{9}{25}} = 1$$

and

$$|-\tfrac{1}{5}\mathbf{u}| = |(-\tfrac{4}{5}, \tfrac{3}{5})| = \sqrt{\tfrac{16}{25} + \tfrac{9}{25}} = 1\,.$$

The same situation is found for three-dimensional vectors. The line segments joining the origin to points on the sphere with centre the origin and radius 1 represent unit vectors so that there is a unit vector in every direction.

Example 2. Find a unit vector in the direction of the vector $\mathbf{u} = (2, -1, 3)$.

Solution:

$$|\mathbf{u}| = \sqrt{4 + 1 + 9} = \sqrt{14}\,.$$

Thus $\frac{1}{\sqrt{14}}\mathbf{u}$ has length 1.

$$\tfrac{1}{\sqrt{14}}\mathbf{u} = \left(\tfrac{2}{\sqrt{14}}, \tfrac{-1}{\sqrt{14}}, \tfrac{3}{\sqrt{14}}\right).$$

$$\left|\tfrac{1}{\sqrt{14}}\mathbf{u}\right| = \sqrt{\tfrac{4}{14} + \tfrac{1}{14} + \tfrac{9}{14}} = 1.$$

$-\frac{1}{\sqrt{14}}\mathbf{u}$ also has length 1 and is collinear with $\mathbf{u}$.

We conclude that $\frac{\pm 1}{|\mathbf{v}|}\mathbf{v}$ are unit vectors collinear with $\mathbf{v}$.

EXERCISE 3.4

1. Find two unit vectors collinear with each of the following vectors.
   (a) $(1, 1)$ (b) $(-3, 1)$ (c) $(-2, -5)$ (d) $(\frac{1}{2}, -\frac{1}{2})$
   (e) $(1, 1, 2)$ (f) $(-1, 3, 1)$ (g) $(2, -2, 3)$ (h) $(-3, -3, 4)$

2. Find unit vectors parallel to $\overrightarrow{PQ}$ for the following P and Q.
   (a) $P(0, 2)$, $Q(1, 3)$ (b) $P(-1, 4)$, $Q(3, -2)$
   (c) $P(-2, -3)$, $Q(-5, -2)$ (d) $P(0, 0, 2)$, $Q(-3, 1, 4)$
   (e) $P(-1, 2, 2)$, $Q(3, 1, -5)$ (f) $P(-3, 1, -3)$, $Q(4, 1, 0)$

3. If $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = 2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, and $\mathbf{w} = \mathbf{u} + 3\mathbf{v}$, find unit vectors collinear with the following.
   (a) $\mathbf{u}$ (b) $\mathbf{u} + \mathbf{v}$ (c) $\mathbf{v} + \mathbf{w}$ (d) $\mathbf{u} + \mathbf{v} + \mathbf{w}$
   (e) $2\mathbf{u} - 3\mathbf{v}$ (f) $3\mathbf{u} + \mathbf{v} + \mathbf{w}$ (g) $\frac{1}{2}\mathbf{u} + \mathbf{v} - 3\mathbf{w}$

3.5. Inner or Dot Product

DEFINITION. If $\mathbf{u} = (a_1, a_2)$ and $\mathbf{v} = (b_1, b_2)$, we define the dot product or inner product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ to be

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2 .$$

If $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{v} = (b_1, b_2, b_3)$, the dot product or inner product is defined similarly to be

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

Thus the inner product of two vectors is simply the sum of the products of corresponding components of the two vectors. The name "scalar product" is also often used for $\mathbf{u} \cdot \mathbf{v}$.

Note that the inner product of two vectors is a scalar (hence the name "scalar product") and not a vector; the set of two-dimensional (three-dimensional) vectors is not closed under this "multiplication". Of course the name "dot product" stems from the symbol $\mathbf{u} \cdot \mathbf{v}$ used to denote the product.

Example 1. Find $\mathbf{u} \cdot \mathbf{v}$ if

(a) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-1, 7)$,

(b) $\mathbf{u} = (3, -1, 5)$, $\mathbf{v} = (-1, -1, 2)$.

Solution:

(a) $\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (-1, 7) = 3(-1) + (-2)7 = -3 - 14 = -17$.

(b) $\mathbf{u} \cdot \mathbf{v} = (3, -1, 5) \cdot (-1, -1, 2) = 3(-1) + (-1)(-1) + 5(2) = 8$.

Example 2. Find k so that $\mathbf{v} \cdot \mathbf{w} = 0$ if $\mathbf{v} = (3, -k, 2)$ and $\mathbf{w} = (-1, 4, 3)$.

Solution:

$$\mathbf{v} \cdot \mathbf{w} = (3, -k, 2) \cdot (-1, 4, 3) = -3 - 4k + 6 = 3 - 4k.$$

Therefore,

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{if} \quad k = \tfrac{3}{4}.$$

Example 3. Find $\mathbf{v} \cdot \mathbf{v}$ if $\mathbf{v} = (-1, 3, -2)$.

Solution:

$$\mathbf{v} \cdot \mathbf{v} = (-1, 3, -2) \cdot (-1, 3, -2) = 1 + 9 + 4 = 14.$$

Example 4. If $\mathbf{u} = (3, -1)$, $\mathbf{v} = (2, -3)$, and $\mathbf{w} = (-1, 5)$, find

(a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$,

(b) $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

Solution:

(a)
$$\mathbf{v} + \mathbf{w} = (2, -3) + (-1, 5) = (1, 2).$$
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (3, -1) \cdot (1, 2) = 3 - 2 = 1.$$

(b)
$$\mathbf{u} \cdot \mathbf{v} = (3, -1) \cdot (2, -3) = 6 + 3 = 9.$$
$$\mathbf{u} \cdot \mathbf{w} = (3, -1) \cdot (-1, 5) = -3 - 5 = -8.$$
$$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = 1.$$

EXERCISE 3.5

1. Calculate $\mathbf{u} \cdot \mathbf{w}$ for the following pairs of vectors.

(a) $\mathbf{u} = (1, 0)$, $\mathbf{w} = (3, 5)$

(b) $\mathbf{u} = (0, 1)$, $\mathbf{w} = (3, 5)$

(c) $\mathbf{u} = (1, -1)$, $\mathbf{w} = (-1, 1)$

(d) $\mathbf{u} = (3, 5)$, $\mathbf{w} = (-1, 4)$

(e) $\mathbf{u} = (\frac{1}{2}, \frac{1}{4})$, $\mathbf{w} = (4, -2)$

(f) $\mathbf{u} = (a, b)$, $\mathbf{w} = (b, a)$

(g) $\mathbf{u} = (a, b)$, $\mathbf{w} = (b, -a)$

(h) $\mathbf{u} = (\pi, \pi)$, $\mathbf{w} = (\frac{1}{\pi}, \frac{1}{\pi})$

(i) $\mathbf{u} = (0, 0, 1)$, $\mathbf{w} = (0, 1, 0)$

(j) $\mathbf{u} = (-1, 0, 1)$, $\mathbf{w} = (1, 1, 1)$

(k) $\mathbf{u} = (2, -1, 4)$, $\mathbf{w} = (3, 3, -1)$

(l) $\mathbf{u} = (-1, 5, -2)$, $\mathbf{w} = (-4, 2, -3)$

(m) $\mathbf{u} = (-1, \frac{1}{2}, 3)$, $\mathbf{w} = (4, 4, -1)$

(n) $\mathbf{u} = (3, -\frac{1}{3}, -1)$, $\mathbf{w} = (\frac{1}{3}, 3, -1)$

(o) $\mathbf{u} = (a, b, a)$, $\mathbf{w} = (b, a, b)$

(p) $\mathbf{u} = (a, a, b)$, $\mathbf{w} = (-a, b, -a)$

2. Find k so that
   (a) $(3, -k) \cdot (5, 4) = 0$, (b) $(2, -4) \cdot (k, -3) = -7$,
   (c) $(-1, k, 2) \cdot (4, 1, -1) = 2$, (d) $(-k, 3, -4) \cdot (2, -1, k) = 6$.
3. Find
   (a) $\mathbf{e}_1 \cdot \mathbf{e}_1$, (b) $\mathbf{e}_2 \cdot \mathbf{e}_2$, (c) $\mathbf{e}_3 \cdot \mathbf{e}_3$.
4. Find
   (a) $\mathbf{e}_1 \cdot \mathbf{e}_2$, (b) $\mathbf{e}_2 \cdot \mathbf{e}_3$, (c) $\mathbf{e}_3 \cdot \mathbf{e}_1$.
5. Find $\mathbf{u} \cdot \mathbf{u}$ if
   (a) $\mathbf{u} = (-3, 4)$, (b) $\mathbf{u} = (3, -2, 1)$, (c) $\mathbf{u} = (a, b, c)$.
6. Given $\mathbf{u} = (3, -2)$, $\mathbf{v} = (4, -7)$, and $\mathbf{w} = (-2, 3)$, find the following.
   (a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ (b) $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (c) $\mathbf{v} \cdot (\mathbf{u} + \mathbf{w})$
   (d) $\mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w}$ (e) $\mathbf{u} \cdot 3\mathbf{v}$ (f) $3(\mathbf{u} \cdot \mathbf{v})$
   (g) $3\mathbf{v} \cdot 4\mathbf{w}$ (h) $12\mathbf{v} \cdot \mathbf{w}$ (i) $-3\mathbf{u} \cdot -4\mathbf{v}$
7. Repeat question (6) if $\mathbf{u} = (4, -1, -1)$, $\mathbf{v} = (-3, 1, 5)$, and $\mathbf{w} = (4, -1, 2)$.

3.6. Properties of the Inner Product. I

In the next three sections, we consider some properties of the inner product. We state these properties and exemplify them and prove them for two-dimensional vectors. The proofs for three-dimensional vectors are left as exercises to the student.

Example 1. Calculate $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ if
(a) $\mathbf{u} = (3, -1)$, $\mathbf{v} = (4, -3)$,
(b) $\mathbf{u} = (-2, 1, 4)$, $\mathbf{v} = (-2, -3, 6)$.

Solution:

(a)
$$\mathbf{u} \cdot \mathbf{v} = 3(4) + (-1)(-3) = 15.$$
$$\mathbf{v} \cdot \mathbf{u} = 4(3) + (-3)(-1) = 15.$$

Therefore,

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for these vectors $\mathbf{u}$ and $\mathbf{v}$.

(b)
$$\mathbf{u} \cdot \mathbf{v} = (-2)(-2) + 1(-3) + 4(6) = 25.$$
$$\mathbf{v} \cdot \mathbf{u} = (-2)(-2) + (-3)(1) + 6(4) = 25.$$

Therefore,

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for these vectors $\mathbf{u}$ and $\mathbf{v}$.

It should be evident that inner multiplication of real vectors involves multiplication of real numbers (the components), and that the order of the components in each product is thus immaterial, so that inner multiplication is commutative.

Property 1 $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

Proof (in V_2): Let $\mathbf{u} = (a_1, a_2)$, $\mathbf{v} = (b_1, b_2)$.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= a_1 b_1 + a_2 b_2 \\ &= b_1 a_1 + b_2 a_2 \\ &= \mathbf{v} \cdot \mathbf{u}. \end{aligned}$$

Therefore,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \text{ for vectors } \mathbf{u} \text{ and } \mathbf{v} \text{ in } V_2.$$

Example 2. Calculate $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ and $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ if

(a) $\mathbf{u} = (3, -1)$, $\mathbf{v} = (-2, 5)$, $\mathbf{w} = (1, -4)$,

(b) $\mathbf{u} = (3, -1, 2)$, $\mathbf{v} = (-2, 5, 3)$, $\mathbf{w} = (1, -4, -4)$.

Solution.

(a) $$\mathbf{v} + \mathbf{w} = (-2, 5) + (1, -4) = (-1, 1).$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (3, -1) \cdot (-1, 1) = -3 - 1 = -4.$$

Also

$$\mathbf{u} \cdot \mathbf{v} = (3, -1) \cdot (-2, 5) = -6 - 5 = -11$$

and

$$\mathbf{u} \cdot \mathbf{w} = (3, -1) \cdot (1, -4) = 3 + 4 = 7.$$

Thus

$$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = -11 + 7 = -4.$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \text{ for these vectors } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w}.$$

(b) $$\mathbf{v} + \mathbf{w} = (-2, 5, 3) + (1, -4, -4) = (-1, 1, -1).$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (3, -1, 2) \cdot (-1, 1, -1) = -3 - 1 - 2 = -6.$$

Also

$$\mathbf{u} \cdot \mathbf{v} = (3, -1, 2) \cdot (-2, 5, 3) = -6 - 5 + 6 = -5$$

and

$$\mathbf{u} \cdot \mathbf{w} = (3, -1, 2) \cdot (1, -4, -4) = 3 + 4 - 8 = -1.$$

Thus

$$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = -5 - 1 = -6.$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \text{ for these vectors } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w}.$$

These examples suggest that inner multiplication is distributive over vector addition.

Property 2 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

Proof (in V_2): Let $\mathbf{u} = (a_1, a_2)$, $\mathbf{v} = (b_1, b_2)$, $\mathbf{w} = (c_1, c_2)$.

$$\mathbf{v} + \mathbf{w} = (b_1, b_2) + (c_1, c_2) = (b_1 + c_1, b_2 + c_2)\,.$$

Therefore,

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (a_1, a_2) \cdot (b_1 + c_1, b_2 + c_2) \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2\,.\end{aligned}$$

Also

$$\mathbf{u} \cdot \mathbf{v} = (a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2$$

and

$$\mathbf{u} \cdot \mathbf{w} = (a_1, a_2) \cdot (c_1, c_2) = a_1c_1 + a_2c_2\,.$$

Therefore,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} &= (a_1b_1 + a_2b_2) + (a_1c_1 + a_2c_2) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2\,.\end{aligned}$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \text{ for vectors } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \text{ in } V_2.$$

EXERCISE 3.6

1. Verify that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for the following pairs of vectors.
   (a) $\mathbf{u} = (3, -1)$, $\mathbf{v} = (4, 2)$
   (b) $\mathbf{u} = (6, -1)$, $\mathbf{v} = (-5, 7)$
   (c) $\mathbf{u} = (4, 2, -3)$, $\mathbf{v} = (1, -1, 4)$
   (d) $\mathbf{u} = (3, -1, -2)$, $\mathbf{v} = (2, -5, 2)$
   (e) $\mathbf{u} = 2\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{v} = (3, 5)$
   (f) $\mathbf{u} = -\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{v} = 4\mathbf{e}_1 + \mathbf{e}_2$
   (g) $\mathbf{u} = 3\mathbf{e}_1 - \mathbf{e}_2 + 5\mathbf{e}_3$, $\mathbf{v} = (4, 1, -1)$
   (h) $\mathbf{u} = -3\mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3$, $\mathbf{v} = 4\mathbf{e}_1 - 2\mathbf{e}_2 - 3\mathbf{e}_3$
   (i) $\mathbf{u} = (\frac{1}{2}, -\frac{3}{2}, \frac{1}{4})$, $\mathbf{v} = (\frac{3}{2}, -\frac{1}{3}, \frac{1}{2})$

2. Verify that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ for the following vectors.
   (a) $\mathbf{u} = (2, -1)$, $\mathbf{v} = (3, -2)$, $\mathbf{w} = (1, 4)$
   (b) $\mathbf{u} = (3, -4)$, $\mathbf{v} = (-1, 1)$, $\mathbf{w} = (5, 2)$
   (c) $\mathbf{u} = (2, -3, 3)$, $\mathbf{v} = (-1, 4, -1)$, $\mathbf{w} = (6, -3, 0)$
   (d) $\mathbf{u} = (-1, 5, 0)$, $\mathbf{v} = (4, -2, -3)$, $\mathbf{w} = (-5, 2, -1)$

3. Prove that
   $$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
   for vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3.

4. Prove that
   $$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$
   for vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3.

3.7. Properties of the Inner Product. II

Example 1. Calculate $(5\mathbf{u}) \cdot \mathbf{v}$, $\mathbf{u} \cdot (5\mathbf{v})$, and $5(\mathbf{u} \cdot \mathbf{v})$ if

(a) $\mathbf{u} = (3, -4)$, $\mathbf{v} = (-1, -3)$,

(b) $\mathbf{u} = (2, -3, 5)$, $\mathbf{v} = (4, -1, -2)$.

Solution:

(a)
$$\begin{aligned} 5\mathbf{u} &= 5(3, -4) = (15, -20). \\ (5\mathbf{u}) \cdot \mathbf{v} &= (15, -20) \cdot (-1, -3) = -15 + 60 = 45. \\ 5\mathbf{v} &= 5(-1, -3) = (-5, -15). \\ \mathbf{u} \cdot 5\mathbf{v} &= (3, -4) \cdot (-5, -15) = -15 + 60 = 45. \\ \mathbf{u} \cdot \mathbf{v} &= (3, -4) \cdot (-1, -3) = -3 + 12 = 9. \\ 5(\mathbf{u} \cdot \mathbf{v}) &= 45. \end{aligned}$$

Therefore $(5\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (5\mathbf{v}) = 5(\mathbf{u} \cdot \mathbf{v})$ for these vectors $\mathbf{u}$ and $\mathbf{v}$.

(b)
$$\begin{aligned} 5\mathbf{u} &= 5(2, -3, 5) = (10, -15, 25). \\ (5\mathbf{u}) \cdot \mathbf{v} &= (10, -15, 25) \cdot (4, -1, -2) = 40 + 15 - 50 = 5. \\ 5\mathbf{v} &= 5(4, -1, -2) = (20, -5, -10). \\ \mathbf{u} \cdot (5\mathbf{v}) &= (2, -3, 5) \cdot (20, -5, -10) = 40 + 15 - 50 = 5. \\ \mathbf{u} \cdot \mathbf{v} &= (2, -3, 5) \cdot (4, -1, -2) = 8 + 3 - 10 = 1. \\ 5(\mathbf{u} \cdot \mathbf{v}) &= 5. \end{aligned}$$

Therefore $(5\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (5\mathbf{v}) = 5(\mathbf{u} \cdot \mathbf{v})$ for these vectors $\mathbf{u}$ and $\mathbf{v}$.

These examples illustrate the following property.

Property 3

$(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$ for any real number k.

Proof (in V_2): Let $\mathbf{u} = (a_1, a_2)$ and $\mathbf{v} = (b_1, b_2)$.
Then

$$\begin{aligned} k\mathbf{u} &= k(a_1, a_2) = (ka_1, ka_2). \\ (k\mathbf{u}) \cdot \mathbf{v} &= (ka_1, ka_2) \cdot (b_1, b_2) \\ &= (ka_1)b_1 + (ka_2)b_2 \\ &= ka_1b_1 + ka_2b_2. \\ k\mathbf{v} &= k(b_1, b_2) = (kb_1, kb_2). \\ \mathbf{u} \cdot (k\mathbf{v}) &= (a_1, a_2) \cdot (kb_1, kb_2) \\ &= a_1(kb_1) + a_2(kb_2) \\ &= ka_1b_1 + ka_2b_2. \\ \mathbf{u} \cdot \mathbf{v} &= (a_1, a_2) \cdot (b_1, b_2) \\ &= a_1b_1 + a_2b_2. \\ k(\mathbf{u} \cdot \mathbf{v}) &= k(a_1b_1 + a_2b_2) = ka_1b_1 + ka_2b_2. \end{aligned}$$

Therefore $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$ for vectors $\mathbf{u}$ and $\mathbf{v}$ in V_2.

Example 2. Calculate $\mathbf{v} \cdot \mathbf{v}$ and $|\mathbf{v}|^2$ for

(a) $|\mathbf{v}| = (3, -1)$,

(b) $|\mathbf{v}| = (-2, 5, 2)$.

Solution:

(a) $$\mathbf{v} \cdot \mathbf{v} = (3, -1) \cdot (3, -1) = 9 + 1 = 10.$$
$$|\mathbf{v}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{10}.$$
$$|\mathbf{v}|^2 = 10.$$

Therefore, $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ for this vector $\mathbf{v}$.

(b) $$|\mathbf{v}| \cdot \mathbf{v} = (-2, 5, 2) \cdot (-2, 5, 2) = 4 + 25 + 4 = 33.$$
$$|\mathbf{v}| = \sqrt{(-2)^2 + 5^2 + 2^2} = \sqrt{33}.$$
$$|\mathbf{v}|^2 = 33.$$

Therefore, $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ for this vector $\mathbf{v}$.

These examples illustrate the following property.

Property 4 $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

Proof (in V_2): Let $\mathbf{v} = (a, b)$.

Then

$$\mathbf{v} \cdot \mathbf{v} = (a, b) \cdot (a, b) = a^2 + b^2.$$

Also

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$
$$|\mathbf{v}|^2 = a^2 + b^2.$$

Therefore, $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ for vectors $\mathbf{v}$ in V_2.

EXERCISE 3.7

1. Verify that $(2\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v})$ for the following vectors.
   (a) $\mathbf{u} = (3, -4)$, $\mathbf{v} = (-6, 2)$
   (b) $\mathbf{u} = (-1, 4)$, $\mathbf{v} = (5, 0)$
   (c) $\mathbf{u} = 2\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{v} = (5, -7)$
   (d) $\mathbf{u} = \mathbf{e}_1 + 3\mathbf{e}_2$, $\mathbf{v} = -\mathbf{e}_1 + 5\mathbf{e}_2$
   (e) $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (-3, 1, 4)$
   (f) $\mathbf{u} = (-3, 1, -5)$, $\mathbf{v} = (-1, 0, 4)$
   (g) $\mathbf{u} = 3\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v} = (4, -1, 3)$
   (h) $\mathbf{u} = 6\mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3$, $\mathbf{v} = 6\mathbf{e}_1 + 4\mathbf{e}_3$
2. Verify that $(-3\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (-3\mathbf{v}) = -3(\mathbf{u} \cdot \mathbf{v})$ for the vectors in question (1).
3. Verify that $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ for the following vectors.
   (a) $\mathbf{v} = (5, -2)$
   (b) $\mathbf{v} = -3\mathbf{e}_1 + \mathbf{e}_2$
   (c) $\mathbf{v} = -4\mathbf{e}_1 + 5\mathbf{e}_2$
   (d) $\mathbf{v} = (-6, 3, 1)$
   (e) $\mathbf{v} = -\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3$
   (f) $\mathbf{v} = 2\mathbf{e}_1 - \mathbf{e}_2 + 5\mathbf{e}_3$
4. Prove Property 3 for vectors in V_3.
5. Prove Property 4 for vectors in V_3.

6. Prove that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{w} + \mathbf{x}) = \mathbf{u} \cdot \mathbf{w} + \mathbf{u} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{x}$.
7. Simplify (a) $(2\mathbf{u} - 3\mathbf{v}) \cdot (\mathbf{u} + 4\mathbf{v})$, (b) $(3\mathbf{u} - 4\mathbf{v}) \cdot (-2\mathbf{u} + 3\mathbf{v})$.
8. Evaluate the vectors in question 7(a) and (b) if $\mathbf{u}$ and $\mathbf{v}$ are vectors of lengths 2 and 3, respectively, and $\mathbf{u} \cdot \mathbf{v} = -6$.

3.8. Inner Product and the Cosine Law

The Cosine Law is an extension of the theorem of Pythagoras and states that in a triangle with sides and angles as shown in Figure 3.4, we have the relation

$$a^2 = b^2 + c^2 - 2bc \cos A \, .$$

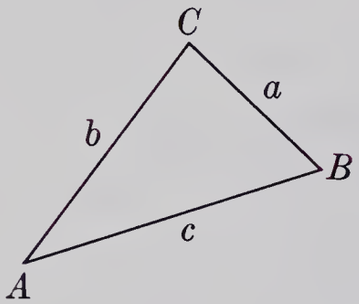


Figure 3.4

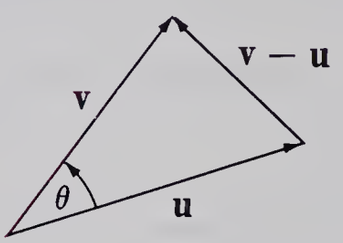


Figure 3.5

In this section we shall apply this law to the vector triangle shown in Figure 3.5 to prove the following result for vectors in V_2 (the proof for vectors in V_3 is left as an exercise).

Theorem. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \, |\mathbf{v}| \cos \theta$,
where θ is the angle between the direction of vectors $\mathbf{u}$ and $\mathbf{v}$, $0 \leq \theta < \pi$.

Proof: Applying the cosine law to this triangle, in which the lengths of the sides are $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{v} - \mathbf{u}|$, yields

$$|\mathbf{v} - \mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2 |\mathbf{u}| \, |\mathbf{v}| \cos \theta \, .$$

Hence,

$$2 |\mathbf{u}| \, |\mathbf{v}| \cos \theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{v} - \mathbf{u}|^2 \, .$$

Now let

$$\mathbf{u} = (a_1, a_2) \text{ and } \mathbf{v} = (b_1, b_2) \, ,$$

so that

$$\mathbf{v} - \mathbf{u} = (b_1 - a_1, b_2 - a_2) \, .$$

Then

$$|\mathbf{u}|^2 = a_1^2 + a_2^2 \, ,$$
$$|\mathbf{v}|^2 = b_1^2 + b_2^2 \, ,$$

and

$$|\mathbf{v} - \mathbf{u}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 \, .$$

Hence,

$$2\,|\,\mathbf{u}\,|\,\,|\,\mathbf{v}\,|\cos\theta = a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2b_2a_2 + a_2^2)$$
$$= 2a_1b_1 + 2a_2b_2\,.$$

Therefore,

$$|\,\mathbf{u}\,|\,\,|\,\mathbf{v}\,|\cos\theta = a_1b_1 + a_2b_2$$
$$= \mathbf{u}\cdot\mathbf{v}\,.$$

Thus the theorem is proved for vectors in V_2.

Example 1. Verify the result of this theorem using the vectors

$$\mathbf{u} = (4, 0) \text{ and } \mathbf{v} = (0, 2)\,.$$

Solution: These vectors may be represented by the line segments joining the origin to the points $A\,(4, 0)$ and $B\,(0, 2)$, respectively, (Figure 3.6). The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{2}$. Thus

$$|\,\mathbf{u}\,|\,\,|\,\mathbf{v}\,|\cos\theta = |\,\mathbf{u}\,|\,\,|\,\mathbf{v}\,|\cos\tfrac{\pi}{2} = 0$$

and

$$\mathbf{u}\cdot\mathbf{v} = (4, 0)\cdot(0, 2) = 0\,.$$

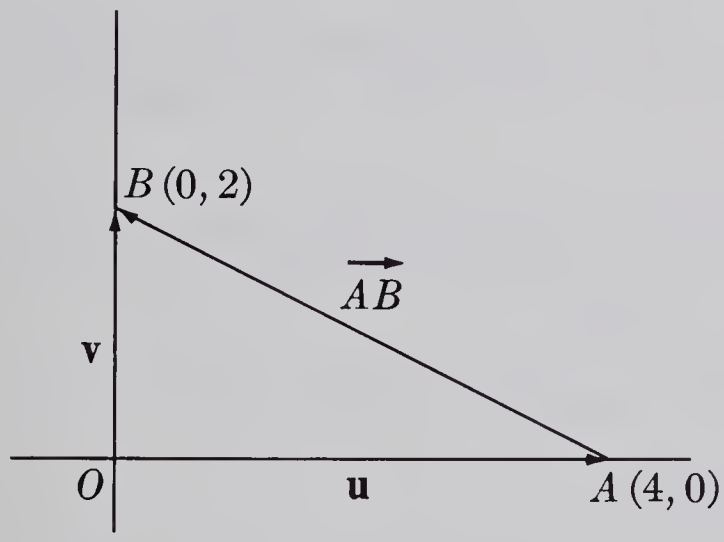


Figure 3.6

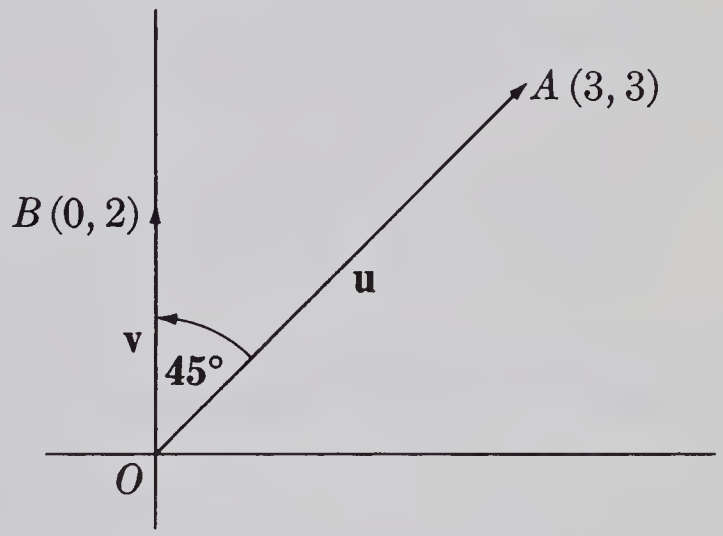


Figure 3.7

Example 2. Verify the result of the theorem using the vectors

$$\mathbf{u} = (3, 3) \text{ and } \mathbf{v} = (0, 2)\,.$$

Solution: The vectors $\mathbf{u}$ and $\mathbf{v}$ are represented geometrically as shown in Figure 3.7. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is 45°. Also $|\,\mathbf{u}\,| = 3\sqrt{2}$ and $|\,\mathbf{v}\,| = 2$. Thus

$$|\,\mathbf{u}\,|\,\,|\,\mathbf{v}\,|\cos\theta = (3\sqrt{2})\,(2)\cos 45^\circ = (3\sqrt{2})\,(2)\,(\tfrac{1}{\sqrt{2}}) = 6$$

and

$$\mathbf{u}\cdot\mathbf{v} = (3, 3)\cdot(2, 0) = 6\,.$$

EXERCISE 3.8

1. Verify that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$ for the following pairs of vectors.
   (a) $(0, -3), (4, 0)$ (b) $(5, 0), (2, 0)$ (c) $(0, 4), (0, -4)$
   (d) $(3, 0), (-2, 0)$ (e) $(2, 5), (4, 10)$ (f) $(1, -3), (-2, 6)$
   (g) $(4, 4), (3, 0)$ (h) $(4, 4), (-3, 0)$ (i) $(4, 4), (0, 3)$
   (j) $(4, 4), (0, -3)$ (k) $(4, 4), (-4, 4)$ (l) $(4, 4), (4, -4)$
2. Repeat question (1) for the following pairs of vectors.
   (a) $(2, 0, 0), (0, 3, 0)$ (b) $(0, 3, 0), (0, -3, 0)$
   (c) $(2, 2, 0), (0, 0, 3)$ (d) $(2, 2, 0), (2, 2, 2\sqrt{2})$
   (e) $(3, 4, 0), (3, 4, 5)$ (f) $(5, -12, 0), (5, -12, 13)$
3. Prove the theorem of this section for vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3.

3.9. Angle Between Vectors

Vectors in V_2 and V_3 may be represented geometrically by line segments with initial points at the origin. Two such line segments representing vectors $\mathbf{u}$ and $\mathbf{v}$ will determine a plane, and in that plane will determine an angle which is defined to be the angle between the algebraic vectors $\mathbf{u}$ and $\mathbf{v}$.

DEFINITION. The angle θ between the two- (three-) dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ is the angle between the line segments with initial points at the origin that represent the two vectors.

From Section 3.8, we see that this angle may be determined from the equation

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta\,.$$

Indeed, if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, we may write

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\,.$$

Example 1. Find, to the nearest degree, the angle between the vectors

$$\mathbf{u} = (3, -1) \text{ and } \mathbf{v} = (3, 7)\,.$$

Solution:

$$\mathbf{u} \cdot \mathbf{v} = (3, -1) \cdot (3, 7) = 3(3) + (-1)(7) = 2\,.$$
$$|\mathbf{u}| = \sqrt{9 + 1} = \sqrt{10}; \quad |\mathbf{v}| = \sqrt{9 + 49} = \sqrt{58}\,.$$

$$\cos\theta = \frac{2}{\sqrt{10}\sqrt{58}} \simeq .0834 \quad \text{(using logarithms)}\,.$$

Therefore, $\theta = 85°$ (to the nearest degree).

Example 2. Find, to the nearest degree, the angle between the vectors

$$\mathbf{u} = (3, -1, 2) \text{ and } \mathbf{v} = (2, 3, -4) .$$

Solution:

$$\mathbf{u} \cdot \mathbf{v} = (3, -1, 2) \cdot (2, 3, -4) = 6 - 3 - 8 = -5 .$$

$$|\mathbf{u}| = \sqrt{9 + 1 + 4} = \sqrt{14} ;$$

$$|\mathbf{v}| = \sqrt{4 + 9 + 16} = \sqrt{29} .$$

$$\cos \theta = \frac{-5}{\sqrt{14}\sqrt{29}} \simeq -0 \cdot 243 \quad \text{(using logarithms)} .$$

Thus, $\theta = 180° - 76° = 104°$ (to the nearest degree).

Example 3. Find the angle between the vectors $\mathbf{u} = (3, -4)$ and $\mathbf{v} = (4, 3)$.

Solution:

$$\mathbf{u} \cdot \mathbf{v} = (3, -4) \cdot (4, 3) = 12 - 12 = 0 .$$

$$|\mathbf{u}| = \sqrt{9 + 16} = 5 ;$$

$$|\mathbf{v}| = \sqrt{16 + 9} = 5 .$$

$$\cos \theta = \frac{0}{5(5)} = 0 .$$

Hence, $\theta = 90°$. The angle between the vectors is, by definition, 90°. The line segments with initial points at the origin that represent $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

DEFINITION. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular or orthogonal if the angle between them is 90°.

It follows that two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. For, in that case,

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = 0 ,$$

and so $\theta = 90°$.

Example 4. Find the value of k such that the vectors $\mathbf{u} = (3, 4, -2)$ and $\mathbf{v} = (5, -1, k)$ are orthogonal.

Solution: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. But

$$\mathbf{u} \cdot \mathbf{v} = (3, 4, -2) \cdot (5, -1, k) = 15 - 4 - 2k .$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \text{ if and only if } 11 - 2k = 0 .$$

$$k = \tfrac{11}{2} .$$

Therefore, $(3, 4, -2)$ and $(5, -1, \frac{11}{2})$ are orthogonal.

Example 5. Find $\angle A$ in $\triangle ABC$ with vertices $A(-3, 1)$, $B(2, 5)$, and $C(-5, 4)$.

Solution: The required $\angle A$ is the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (Figure 3.8).

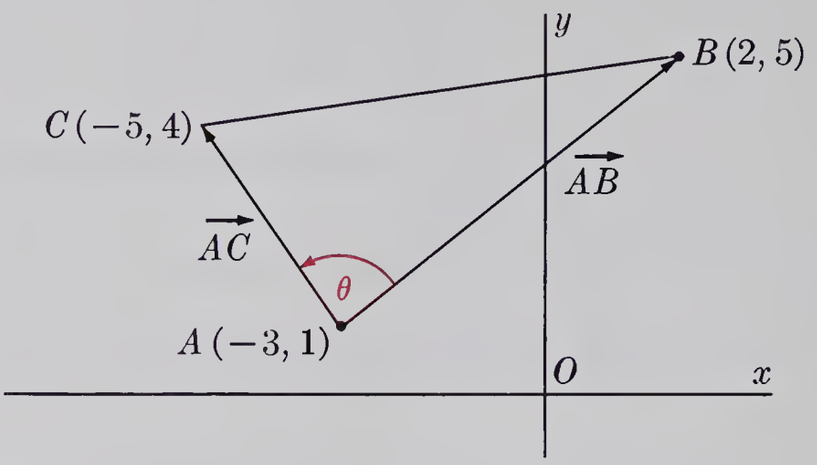


Figure 3.8

$$\overrightarrow{AB} = (2 - (-3), 5 - 1) = (5, 4).$$
$$\overrightarrow{AC} = (-5 - (-3), 4 - 1) = (-2, 3).$$
$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (5, 4) \cdot (-2, 3) = -10 + 12 = 2.$$
$$|\overrightarrow{AB}| = \sqrt{25 + 16} = \sqrt{41}; \quad |\overrightarrow{AC}| = \sqrt{4 + 9} = \sqrt{13}.$$
$$\cos\theta = \frac{2}{\sqrt{41}\sqrt{13}} \simeq .0865 \quad \text{(using logarithms)}.$$
$$\theta = 85° \quad \text{(to the nearest degree)}.$$

Note that, from the diagram, this seems to be a reasonable answer. Triangles sketched in three dimensions may not provide as useful an indication to the validity of the result. Note also that logarithms were used to obtain an approximation for $\frac{2}{\sqrt{41}\sqrt{13}}$. The validity of this calculation can be checked by approximating

$$\frac{2}{\sqrt{41}\sqrt{13}} \simeq \frac{2}{6 \times 4} = \frac{1}{12} \simeq .08.$$

EXERCISE 3.9

1. Find the cosine of the angle between the following vectors **u** and **v**.
   (a) $(3, 4), (-1, 5)$ (b) $(-5, 2), (6, -3)$
   (c) $(4, 1), (-7, 0)$ (d) $(3, -1, -1), (4, 2, 1)$
   (e) $(3, -1, 0), (4, 0, -3)$ (f) $(3, -2, -5), (-5, 1, 1)$
2. Find to the nearest degree the angle between the following vectors **u** and **v**.
   (a) $(-1, 4), (3, -1)$ (b) $(4, 0), (-1, 3)$
   (c) $(3, -3), (5, -2)$ (d) $(-1, 1, 4), (3, 1, 0)$
   (e) $(0, 5, -2), (-3, 1, 1)$ (f) $(4, -3, -1), (2, 7, 0)$

3. Find to the nearest degree the angle the line segment PQ makes with the positive x-axis if P and Q are the following points.
   (a) $P(3, 4), Q(7, 9)$
   (b) $P(-5, 1), Q(7, 3)$
   (c) $P(-1, -4), Q(-3, 1)$
   (d) $P(1, 1, 4), Q(0, 6, -3)$
   (e) $P(4, 4, -2), Q(-1, 5, 2)$
   (f) $P(3, 0, -2), Q(5, 0, -2)$

4. Find, to the nearest degree,
   (a) the angle the line segment from $P(4, -1)$ to $Q(5, -2)$ makes with the positive y-axis,
   (b) the angle the line segment from $P(4, -1, 3)$ to $Q(5, 0, -3)$ makes with the positive y-axis,
   (c) the angle the line segment from $P(2, -2, 1)$ to $Q(5, -3, 2)$ makes with the positive z-axis.

5. Find the angles of $\triangle PQR$ for $P(2, -1)$, $Q(3, 7)$, and $R(-3, 1)$.

6. Find the angles of $\triangle PQR$ for $P(3, -1, 1)$, $Q(-1, 7, 4)$, and $R = (1, 1, 3)$.

7. For what values of k are the following pairs of vectors orthogonal?
   (a) $(5, 7), (-2, 3k)$
   (b) $6\mathbf{e}_1 - \mathbf{e}_2, k\mathbf{e}_1 + 3\mathbf{e}_2$
   (c) $(4, -1, 3), (2, -k, 5)$
   (d) $3\mathbf{e}_1 - \mathbf{e}_2 + 2k\mathbf{e}_3, 5\mathbf{e}_1 - 6\mathbf{e}_2 + \mathbf{e}_3$.
   (e) $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ for $P(-1, k)$, $Q(4, -3)$, $R(5, -2)$, and $S(0, -3)$.
   (f) $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ for $P(-3, -2, -2k)$, $Q(1, 5, -2)$, $R(0, 3, -1)$, and $S(2, 2, 5)$.

8. Is $\triangle PQR$ right-angled for $P(6, 5)$, $Q(1, 3)$, $R(3, -2)$?

9. Is $\triangle PQR$ right-angled for $P(-1, 3, 2)$, $Q(0, 3, 4)$, $R(2, 0, -1)$?

10. Find h and k so that $\mathbf{u} = (5, h, k)$ is orthogonal to both $\mathbf{v} = (1, 3, 2)$ and $\mathbf{w} = (-2, 1, 4)$.

11. Prove that the angle in a semicircle is a right angle. (Hint: in Figure 3.9 show that $\overrightarrow{BC} \cdot \overrightarrow{CA} = 0$.)

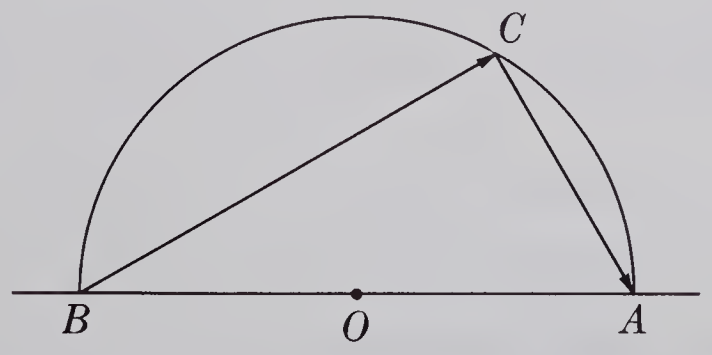


Figure 3.9

3.10. Projections

Consider $\triangle PQR$ shown in Figure 3.10.

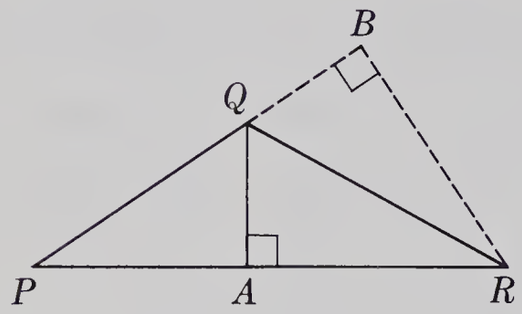


Figure 3.10

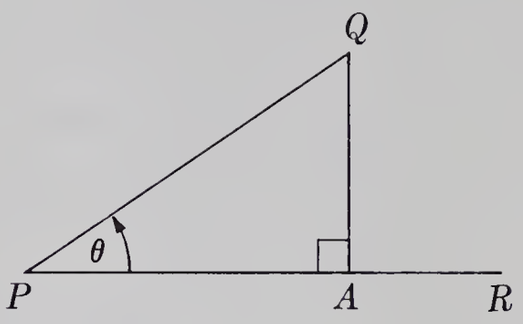


Figure 3.11

Let QA be constructed so that $QA \perp PR$. Then we say that

(the length of) PA is the projection of PQ on PR.

Similarly,

(the length of) AR is the projection of QR on PR.

Finally, if we construct RB so that $RB \perp PQ$, we say that

(the length of) PB is the projection of PR on PQ produced.

In general, we may speak of the projection of any line segment PQ on a coterminal line segment PR; this will be PA where $QA \perp PR$ (Figure 3.11). Note that if θ is the angle between line segments PQ and PR we have

$$\text{projection of } PQ \text{ on } PR = PA = PQ\,|\cos\theta|\,.$$

We may now define the projection of vector **u** on vector **v**.

DEFINITION. The projection of vector **u** on vector **v** is the projection of the line segment from the origin representing **u** on the line segment from the origin representing **v**.

The projection of vector **u** on vector **v** is a nonnegative scalar.

Example 1. Find the projection of the vector $\mathbf{u} = (1, 4)$ on the vector $\mathbf{v} = (4, 3)$.

Solution: By definition, the projection of the vector **u** on the vector **v** will be the projection of line segment OP on line segment OQ for the points $P(1, 4)$ and $Q(4, 3)$ (see Figure 3.12).

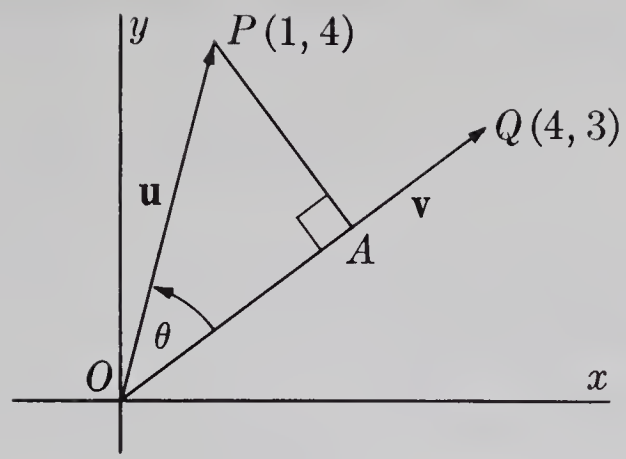


Figure 3.12

Now construct $PA \perp OQ$. The required projection is $OA = OP \cos\theta$.

$$OP = |\mathbf{u}| = \sqrt{1 + 16} = \sqrt{17}$$

and θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$, so that

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{(1, 4) \cdot (4, 3)}{\sqrt{17}\sqrt{25}} = \frac{16}{5\sqrt{17}}.$$

Thus

$$OA = \sqrt{17} \cdot \tfrac{16}{5\sqrt{17}} = \tfrac{16}{5}.$$

Theorem. The projection of vector $\mathbf{u}$ on vector $\mathbf{v}$ is $\dfrac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{v}|}$.

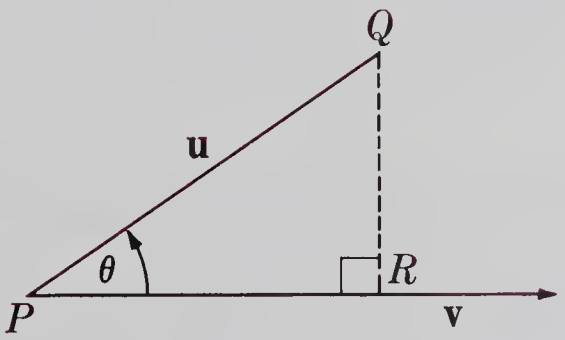


Figure 3.13

Proof: Choosing representatives of $\mathbf{u}$ and $\mathbf{v}$ that are coterminal and assuming angle θ is between the vectors (Figure 3.13), we may state the following.

$$\begin{aligned}
\text{Projection of } \mathbf{u} \text{ on } \mathbf{v} &= PR \\
&= PQ\,|\cos\theta| \\
&= |\mathbf{u}|\,|\cos\theta| \\
&= |\mathbf{u}|\,\frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} \\
&= \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{v}|} \quad \text{as required.}
\end{aligned}$$

Example 2. We may now state immediately that, in Example 1, the projection of $\mathbf{u} = (1, 4)$ on $\mathbf{v} = (4, 3)$ is

$$\frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{v}|} = \frac{|(1, 4) \cdot (4, 3)|}{\sqrt{16 + 9}} = \frac{16}{5}.$$

Example 3. Find the projection of vector $\mathbf{u} = (3, -1, 2)$ on vector $\mathbf{v} = (2, 5, -2)$.

Solution: The required projection is

$$\frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{v}|} = \frac{|(3, -1, -2) \cdot (2, 5, 2)|}{\sqrt{4 + 25 + 4}}$$

$$= \frac{|-3|}{\sqrt{33}}.$$

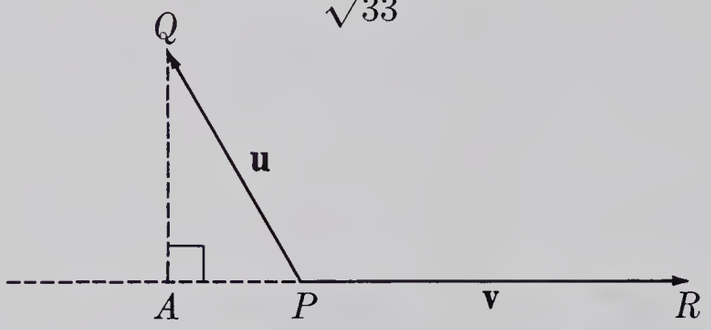


Figure 3.14

The fact that $\mathbf{u} \cdot \mathbf{v} = -3$ suggests a situation as shown in Figure 3.14. The projection PA is measured in the direction opposite to that of $\mathbf{v}$. The angle between $\mathbf{u}$ and $\mathbf{v}$ is greater than 90°.

EXERCISE 3.10

1. Find the projection of vector $\mathbf{u}$ on vector $\mathbf{v}$ for the following pairs of vectors $\mathbf{u}$ and $\mathbf{v}$.
   (a) $\mathbf{u} = (3, 0)$, $\mathbf{v} = (2, 3)$
   (b) $\mathbf{u} = (-1, 5)$, $\mathbf{v} = (4, 3)$
   (c) $\mathbf{u} = (4, 5)$, $\mathbf{v} = (-5, 4)$
   (d) $\mathbf{u} = (6, 1)$, $\mathbf{v} = (-1, 5)$
   (e) $\mathbf{u} = 2\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{v} = (6, 2)$
   (f) $\mathbf{u} = -3\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2$
   (g) $\mathbf{u} = (1, 1, -4)$, $\mathbf{v} = (2, -1, 3)$
   (h) $\mathbf{u} = (4, 2, -1)$, $\mathbf{v} = (3, -1, 2)$
   (i) $\mathbf{u} = (-4, 1, 3)$, $\mathbf{v} = (1, -2, 2)$
   (j) $\mathbf{u} = (3, -3, 1)$, $\mathbf{v} = (2, -1, 4)$
   (k) $\mathbf{u} = 3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{v} = (-3, 1, 2)$
   (l) $\mathbf{u} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{v} = 3\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$
2. Find the projection of vector $\overrightarrow{PQ}$ on vector $\overrightarrow{RS}$ for the following sets of points.
   (a) $P(-3, 1)$, $Q(2, -5)$, $R(4, 2)$, $S(5, -3)$
   (b) $P(-2, 1, 6)$, $Q(3, 1, -2)$, $R(-3, 1, 4)$, $S(2, -1, 2)$
3. When will the projection of $\mathbf{u}$ on $\mathbf{v}$ be zero?

4. Is the projection of $\mathbf{u}$ on $\mathbf{v}$ (a) always, (b) ever, the same as the projection of $\mathbf{v}$ on $\mathbf{u}$? In the event of a yes answer in (b), specify when.

3.11. Resolution of Vectors

Two plane vectors $\mathbf{u}$ and $\mathbf{v}$ can be added to produce a single plane vector $\mathbf{w}$. Similarly, a plane vector $\mathbf{w}$ can be expressed in an infinite number of ways as the sum of two plane vectors $\mathbf{u}$ and $\mathbf{v}$. The process of expressing $\mathbf{w}$ in the form $\mathbf{u} + \mathbf{v}$ is called the resolution of a vector; the vectors $\mathbf{u}$ and $\mathbf{v}$ are called components of $\mathbf{w}$.

The vector $\mathbf{w} = (a, b)$ can be expressed as the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 = (a, 0) + (0, b) ;$$

here the components $\mathbf{w}_1$ and $\mathbf{w}_2$ of $\mathbf{w}$ are vectors collinear with the x- and y-axes, respectively. We may also use the symbols $\mathbf{w}_x$ and $\mathbf{w}_y$ for these components. This is an example of the most important case of the resolution of a vector, namely, its expression as the sum of two perpendicular vectors; in this case the two vectors are called rectangular components.

Example 1. Find rectangular components for the vector $(-2, \frac{5}{2})$.

Solution: One solution is $(-2, \frac{5}{2}) = (-2, 0) + (0, \frac{5}{2})$.

Suppose the vector $\mathbf{w}$ is resolved into rectangular components $\mathbf{w}_1$ and $\mathbf{w}_2$ with $\mathbf{w}_1$ collinear with the vector $\mathbf{u}$; then $\mathbf{w}_2$ is collinear with vector $\mathbf{v}$ which is such that $\mathbf{u} \cdot \mathbf{v} = 0$ (since $\mathbf{u}$ and $\mathbf{v}$ are perpendicular).

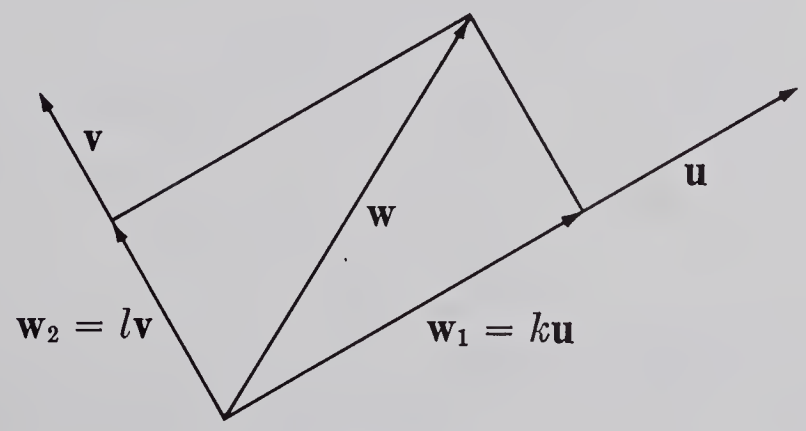


Figure 3.15

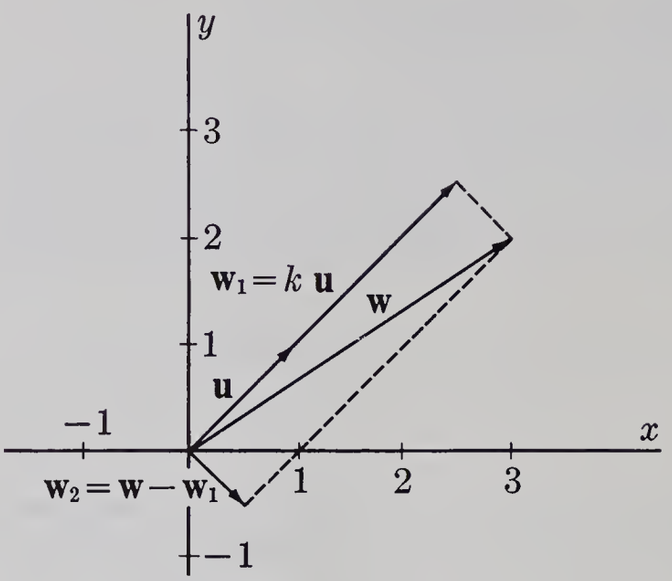
Then (Figure 3.15) we have

$$\mathbf{w}_1 = k\mathbf{u} \quad (\mathbf{w}_1 \text{ is in the direction of } \mathbf{u}) ,$$
$$\mathbf{w}_2 = l\mathbf{v} \quad (\mathbf{w}_2 \text{ is in the direction of } \mathbf{v}) .$$

Therefore,

$$|\mathbf{w}|^2 = |\mathbf{w}_1|^2 + |\mathbf{w}_2|^2 \quad \text{(Theorem of Pythagoras)} .$$

Example 2. Find rectangular components $\mathbf{w}_1$ and $\mathbf{w}_2$ for the vector $\mathbf{w} = (3, 2)$ such that $\mathbf{w}_1$ is collinear with $\mathbf{u} = (1, 1)$.



Solution:

(i) $\mathbf{w} = (3, 2) = \mathbf{w}_1 + \mathbf{w}_2$.

(ii) $\mathbf{w}_1 = k\mathbf{u}$
$= k(1, 1)$
$= (k, k)$.

(iii) $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$.

From (i),

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{w} - \mathbf{w}_1 \\ &= (3, 2) - (k, k) \\ &= (3 - k, 2 - k). \end{aligned}$$

From (iii),

$$\begin{aligned} \mathbf{w}_1 \cdot \mathbf{w}_2 &= (k, k) \cdot (3 - k, 2 - k) \\ &= 3k - k^2 + 2k - k^2 \\ &= 5k - 2k^2 \\ &= 0. \end{aligned}$$

Therefore,

$$k = 0 \text{ or } \tfrac{5}{2}.$$

Obviously, $k \neq 0$ and so $k = \frac{5}{2}$. Therefore,

$$\mathbf{w}_1 = (\tfrac{5}{2}, \tfrac{5}{2}) \text{ and } \mathbf{w}_2 = (\tfrac{1}{2}, -\tfrac{1}{2})$$

are the required rectangular components.

We may also solve this problem by finding $|\mathbf{w}_1|$ as the projection of $\mathbf{w}$ on $\mathbf{u}$.

We may extend this work on resolution of forces to three dimensions. In order to avoid many of the complications that arise, we shall confine our remarks to the resolution of a three-dimensional vector into components collinear with the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, that is, parallel to the co-ordinate axes. We may refer to such components of a vector $\mathbf{w}$ either by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, or by $\mathbf{w}_x$, $\mathbf{w}_y$, and $\mathbf{w}_z$. Note that these components are mutually orthogonal vectors.

Example 3. Resolve the vector $\mathbf{w} = (3, -2, 5)$ into components collinear with $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

Solution:

$$\mathbf{w} = (3, -2, 5) = 3\mathbf{e}_1 - 2\mathbf{e}_2 + 5\mathbf{e}_3 .$$

Therefore,

$$\mathbf{w}_1 = 3\mathbf{e}_1, \quad \mathbf{w}_2 = -2\mathbf{e}_2, \quad \mathbf{w}_3 = 5\mathbf{e}_3 .$$

Example 4. A force of 12 pounds is exerted on an object in the direction of the line segment joining $P(3, -1, 5)$ to $Q(7, 3, 7)$. Find the magnitude of the force parallel to each of the co-ordinate axes.

Solution: Let the force be represented by vector $\mathbf{f}$. Then

$$\begin{aligned} \mathbf{f} &= k(7 - 3, 3 + 1, 7 - 5) \\ &= k(4, 4, 2) . \end{aligned}$$

Also

$$\begin{aligned} |\mathbf{f}| &= k\sqrt{16 + 16 + 4} = 6k . \\ 6k &= 12 \quad \text{and} \quad k = 2 . \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{f} &= 2(4, 4, 2) \\ &= (8, 8, 4) \\ &= 8\mathbf{e}_1 + 8\mathbf{e}_2 + 4\mathbf{e}_3 . \\ \mathbf{f}_x &= 8\mathbf{e}_1, \quad \mathbf{f}_y = 8\mathbf{e}_2, \quad \mathbf{f}_z = 4\mathbf{e}_3 . \end{aligned}$$

The magnitudes of the forces parallel to the x-, y-, and z-axes are 8, 8, and 4 pounds, respectively.

EXERCISE 3.11

1. Find rectangular components $\mathbf{w}_1$ and $\mathbf{w}_2$ parallel to the x- and y-axes, respectively, for the following vectors.
   (a) $(5, \frac{1}{2})$ (b) $(-4, 7)$ (c) $(6, -2)$ (d) $(-3, -4)$
2. The vector $(3, -7)$ is resolved into two rectangular components with equal magnitude; find this magnitude.
3. Repeat question (2) for the vectors (a) $(-2, 6)$, (b) $(-3, -5)$.
4. The vector $(4, -3)$ is resolved into two rectangular components $\mathbf{w}_1$ and $\mathbf{w}_2$ with $|\mathbf{w}_1| = 2|\mathbf{w}_2|$. Find $|\mathbf{w}_2|$.

5. Find $\mathbf{w}_1$ and $\mathbf{w}_2$ such that
$$(3, -5) = \mathbf{w}_1 + \mathbf{w}_2 .$$

6. Find $\mathbf{w}_1$ and $\mathbf{w}_2$ such that
$$(3, -5) = \mathbf{w}_1 + \mathbf{w}_2 \quad \text{and} \quad \mathbf{w}_1 \cdot \mathbf{w}_2 = 0.$$

7. Find $\mathbf{w}_1$ and $\mathbf{w}_2$ such that
$$(3, -5) = \mathbf{w}_1 + \mathbf{w}_2 , \quad \mathbf{w}_1 \cdot \mathbf{w}_2 = 0 , \quad \text{and} \quad \mathbf{w}_1 = k(1, 2) .$$

8. Find rectangular components for $\mathbf{w} = (-3, 4)$, with one of the components parallel to $\mathbf{u} = (-1, 2)$.

9. Resolve $\mathbf{w} = (5, -1)$ into rectangular components, one of which is parallel to $\mathbf{u} = (3, -2)$.

10. A boy is pulling a sled up a hill that has a rise of one foot in seven. The magnitude (pounds) and direction of the force exerted by the boy are represented by the vector $(9, 12)$. Find the force which draws the sled up the hill and the force which tends to lift the sled.

11. Cubesville is one mile due east of Squaresville. Its altitude is 264 feet greater than that of Squaresville and the towns are connected by a smooth-rising road. A disabled Stanley Steamer is being towed from Squaresville to Cubesville by a tow truck exerting a force whose magnitude and direction are represented by the vector $(2000, 1000)$. Find the force drawing the Stanley Steamer up the hill and the force tending to lift it.

12. State components $\mathbf{w}_x$, $\mathbf{w}_y$, and $\mathbf{w}_z$ for the following vectors $\mathbf{w}$.
(a) $(-1, 5, 3)$ (b) $(6, -4, 2)$ (c) $(-1, -6, 5)$

13. A force of 24 pounds is exerted on an object in the direction of the line segment joining point $P(4, -1, 3)$ to $Q(5, 3, -4)$. Find the magnitude of the force parallel to each of the co-ordinate axes.

14. Repeat question (13) if a force of 15 pounds is exerted in the direction of the line segment joining $P(5, -3, -2)$ to $Q(7, -1, 3)$.

15. Resolve vector $\mathbf{w} = (3, -1, 5)$ into components collinear with
$$\mathbf{v}_1 = (-2, 2, 0), \quad \mathbf{v}_2 = (1, 1, 2), \quad \mathbf{v}_3 = (1, 1, -1) .$$
Show that these components are mutually orthogonal vectors.

3.12. Work

When a force is exerted over a certain distance, work is performed and the work done is given by the product

work = (magnitude of force in the direction of motion) (distance).

This concept may be considered in the light of our theory of algebraic vectors. Here, the force is expressed as an algebraic vector such as $\mathbf{w} = (a, b)$; such a vector has, of course, a magnitude and a direction. We consider that such a force is applied to an object and that the object moves a certain distance in a certain direction. The motion of the object can thus be expressed as a second vector $\mathbf{v} = (c, d)$.

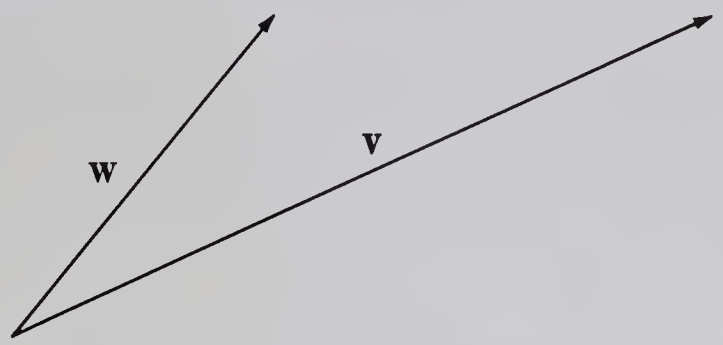


Figure 3.16

Figure 3.16 represents the force $\mathbf{w}$ acting on the object and the path $\mathbf{v}$ along which the object moves under the exertion of this force. Then,

(i) the magnitude of the force $\mathbf{w}$ in the direction of vector $\mathbf{v}$ is the projection of $\mathbf{w}$ on $\mathbf{v}$; that is, from Section 3.10,

$$\frac{|\mathbf{w} \cdot \mathbf{v}|}{|\mathbf{v}|};$$

(ii) the distance the object moves is the magnitude of vector $\mathbf{v}$, that is,

$$|\mathbf{v}|.$$

Therefore the work done by force $\mathbf{w}$ moving an object in the distance and direction represented by vector $\mathbf{v}$ is

$$\frac{|\mathbf{w} \cdot \mathbf{v}|}{|\mathbf{v}|} \times |\mathbf{v}| = |\mathbf{w} \cdot \mathbf{v}|.$$

Example 1. A force (in pounds) expressed by the vector $\mathbf{w} = (3, 7)$ moves an object a distance (in feet) and direction represented by the vector $\mathbf{v} = (2, 9)$. Find the work performed.

Solution:

$$\begin{aligned} \text{Work} &= |\mathbf{w} \cdot \mathbf{v}| \\ &= |(3, 7) \cdot (2, 9)| \\ &= 69. \end{aligned}$$

The work performed is 69 foot-pounds.

Example 2. Find the work done if the force in Example 1 moves an object along vector $\mathbf{v} = (-2, -9)$.

Solution:

$$\begin{aligned} \text{Work} &= |\mathbf{w} \cdot \mathbf{v}| \\ &= (3, 7) \cdot (-2, -9) = 69 . \end{aligned}$$

The negative sign of $\mathbf{w} \cdot \mathbf{v}$ indicates that we may consider that the work exerted in moving an object from point A to point B is the negative of the work exerted in moving an object from point B to point A, or that the work exerted in lowering an object a certain distance may be considered the negative of the amount of work required to raise the object the same distance. Normally we consider that the work is the same in both cases, that is, we take the absolute value as the work performed.

Example 3. A force of 10 pounds acting in a direction parallel to the line segment AB, with $A(1, 1, 2)$ and $B(2, 1, 5)$, is applied to an object as it moves on line segment PQ, with $P(3, -2, 4)$ and $Q(5, 1, -3)$. Find the work done. (Assume that the unit of measurement in the co-ordinate system is the foot.)

Solution: We must first find the vector $\mathbf{w}$ describing this force. The force is parallel to line segment AB, and hence is parallel to vector

$$\overrightarrow{AB} = (2 - 1, 1 - 1, 5 - 2) = (1, 0, 3) .$$

Therefore,

$$\mathbf{w} = k(1, 0, 3)$$

and

$$\begin{aligned} |\mathbf{w}| &= k\sqrt{10} . \\ k\sqrt{10} &= 10 . \\ k &= \sqrt{10} . \end{aligned}$$

Therefore,

$$\mathbf{w} = \sqrt{10}(1, 0, 3) = (\sqrt{10}, 0, 3\sqrt{10}) .$$

Secondly, we find the vector $\mathbf{v}$ over which this force is exerted.

$$\mathbf{v} = \overrightarrow{PQ} = (5 - 3, 1 + 2, -3 - 4) = (2, 3, -7).$$

Then

$$\begin{aligned} |\mathbf{w} \cdot \mathbf{v}| &= |(\sqrt{10}, 0, 3\sqrt{10}) \cdot (2, 3, -7)| \\ &= |2\sqrt{10} - 21\sqrt{10}| \\ &= 19\sqrt{10} , \end{aligned}$$

so that the work done is $19\sqrt{10}$ foot-pounds.

EXERCISE 3.12

1. Find the work done by force $\mathbf{w}$ exerted on an object moving along vector $\mathbf{v}$ if $\mathbf{w}$ and $\mathbf{v}$ are the following.
   (a) $\mathbf{w} = (5, 2)$, $\mathbf{v} = (3, -2)$
   (b) $\mathbf{w} = (5, 2)$, $\mathbf{v} = (-3, 2)$
   (c) $\mathbf{w} = (\frac{1}{2}, \frac{2}{3})$, $\mathbf{v} = (6, -6)$
   (d) $\mathbf{w} = (-3, 0)$, $\mathbf{v} = (0, 2)$
   (e) $\mathbf{w} = (1, 1, 3)$, $\mathbf{v} = (3, 2, 4)$
   (f) $\mathbf{w} = (-3, 1, 4)$, $\mathbf{v} = (2, -2, 3)$
2. Explain the answer in question (1d).
3. A force of 5 pounds acting in the direction of line segment AB, with $A(2, 5)$, and $B(5, 9)$, is exerted on an object moving from point $P(-2, 1)$ to point $Q(3, 7)$ (distance in feet). Find the work done.
4. A force of 19 pounds acting in the direction of line segment EF, with $E(3, 1, 1)$ and $F(-2, -1, 4)$, is exerted on an object moving along PQ, with $P(3, -1, 4)$ and $Q(-2, 5, -1)$ (distance in feet). Find the work done in moving from P to Q.
5. Find the work done by the tow truck in question (11), Exercise 3.11, in towing the Stanley Steamer from Squaresville to Cubesville.

Chapter Summary

Definition of distance $d(P, Q)$ between two points P and Q · Length of line segment PQ · Line segments in 3-space · Length of a vector; unit vector · Inner product (dot product, scalar product)

Properties of the inner product:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$$
$$(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v}).$$
$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2.$$

Angle between two vectors:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

Orthogonal vectors · Projections · Resolution of vectors · Work

REVIEW EXERCISE 3

1. Use the distance formula to determine which of the following sets of points are collinear.
   (a) $P(3, -1, 1)$, $Q(3, -2, 1)$, $R(-2, 1, 2)$
   (b) $P(2, -1, -1)$, $Q(-5, -3, 1)$, $R(-1, -5, -1)$
   (c) $P(3, 4, -1)$, $Q(-1, 2, 3)$, $R(1, 0, -7)$

2. Use the distance formula to determine which of the following triangles PQR are right-angled.
   (a) $P(-1, 3, 2)$, $Q(3, -2, 4)$, $R(5, -2, 0)$
   (b) $P(1, 0, 2)$, $Q(0, 3, 2)$, $R(-3, 1, 0)$
   (c) $P(3, 1, 1)$, $Q(-4, 1, 5)$, $R(0, 3, 12)$
3. Rework questions (1) and (2) using vectors.
4. Find the length of the following vectors.
   (a) $(5, -2)$ (b) $(-3, 1, 1)$ (c) $(4, 1, -2)$
   (d) $\overrightarrow{PQ}$; $P(6, -1)$, $Q(-3, 2)$ (e) $\overrightarrow{PQ}$; $P(-3, 1, -1)$, $Q(-5, -1, 4)$
5. Find unit vectors collinear with the vectors in question (4).
6. If $\mathbf{u} = (-1, 5, 2)$, $\mathbf{v} = -3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$, and $\mathbf{w} = \mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3$, find
   (a) $|\mathbf{u}|$, (b) $|\mathbf{u} + \mathbf{v}|$, (c) $|\mathbf{v} - \mathbf{w}|$, (d) $|2\mathbf{u} - \mathbf{v} + 3\mathbf{w}|$.
7. For the vectors of question (6), calculate the following.
   (a) $\mathbf{u} \cdot \mathbf{v}$ (b) $\mathbf{v} \cdot \mathbf{w}$ (c) $\mathbf{w} \cdot \mathbf{u}$ (d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
   (e) $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (f) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w})$ (g) $\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w}$
   (h) $\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$ (i) $(-4\mathbf{u}) \cdot \mathbf{w}$ (j) $(3\mathbf{u} - \mathbf{v}) \cdot (2\mathbf{v} + \mathbf{w})$
8. For the vectors of question (6), find the cosine of the angle between
   (a) $\mathbf{u}$ and $\mathbf{v}$, (b) $\mathbf{v}$ and $\mathbf{w}$, (c) $\mathbf{w}$ and $\mathbf{u}$.
9. Use vectors to determine the angles of the following triangles PQR.
   (a) $P(-1, 1)$, $Q(3, -1)$, $R(2, 4)$ (b) $P(0, 0, 1)$, $Q(4, -2, 3)$, $R(3, 3, 2)$
10. Find to the nearest degree the angle between $\overrightarrow{PQ}$ and the positive x-axis.
    (a) $P(-1, 4)$, $Q(3, 7)$ (b) $P(2, 2, 5)$, $Q(-1, 5, 4)$
11. Find the projection of vector $\mathbf{u}$ on vector $\mathbf{v}$ for the following vectors.
    (a) $\mathbf{u} = (5, -2)$, $\mathbf{v} = (4, 7)$ (b) $\mathbf{u} = (-1, 2, 1)$, $\mathbf{v} = (3, -4, 4)$
12. Find the work done by force $\mathbf{w}$ exerted on an object moving along vector $\mathbf{v}$ for the following.
    (a) $\mathbf{w} = (3, 1)$, $\mathbf{v} = (2, -6)$ (b) $\mathbf{w} = (2, 1, 0)$, $\mathbf{v} = (-3, 1, 4)$
13. A force of 6 pounds acting in the direction of line segment AB is exerted on an object moving from point P to point Q (distance in feet). Find the work done for the following situations.
    (a) $A(4, 2)$, $B(-3, 6)$, $P(2, 3)$, $Q(5, 7)$
    (b) $A(1, 1, 3)$, $B(3, -4, 2)$, $P(1, 5, -3)$, $Q(3, 4, 6)$
14. A force of 9 pounds is exerted on an object in the direction of the line segment joining point $P(-3, 5, 1)$ to point $Q(4, 3, 2)$. Find the magnitude of the components of the force parallel to the co-ordinate axes.
15. Find the components of the vector $\mathbf{w} = (3, 3, 7)$ collinear with the vectors
    $\mathbf{v}_1 = (1, -3, 0)$, $\mathbf{v}_2 = (3, 1, 0)$, and $\mathbf{v}_3 = (0, 0, 4)$.

Chapter 4

LINES IN TWO- AND THREE-SPACE

4.1. The Vector Equation of a Line

Let a line l be determined by two points A and B and let P be an arbitrary point on the line. Then $\overrightarrow{AP} = k\overrightarrow{AB}$, $k \in Re$, and we have the following possibilities.

Value of k	Position of P if $\overrightarrow{AP} = k\overrightarrow{AB}$
0	at A (Figure 4.1 (a))
1	at B (Figure 4.1 (b))
$0 < k < 1$	between A and B (Figure 4.1 (c))
$k > 1$	B between A and P (Figure 4.1 (d))
$k < 0$	A between P and B (Figure 4.1 (e))

Then $\overrightarrow{AP} = k\overrightarrow{AB}$ represents the following geometric figures.

(i) a point on line AB if $k =$ any specific real number
(ii) line segment AB for $\{k \mid 0 \leq k \leq 1,\ k \in Re\}$
(iii) ray AB for $\{k \mid k \geq 0,\ k \in Re\}$
(iv) ray BA for $\{k \mid k \leq 1,\ k \in Re\}$
(v) line AB for $\{k \mid k \in Re\}$

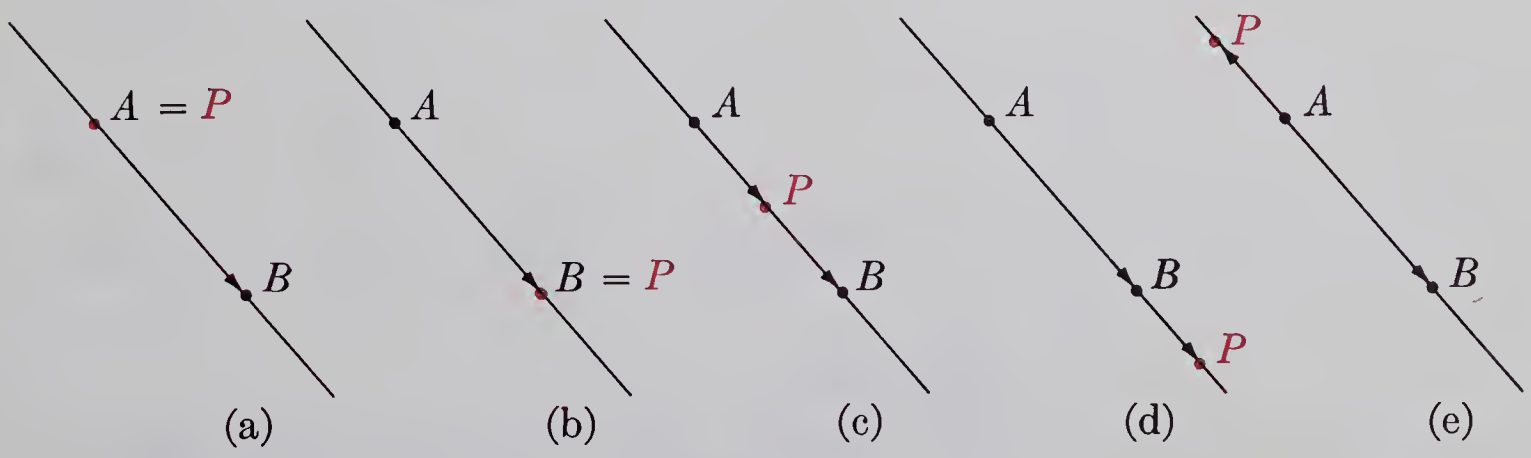


Figure 4.1

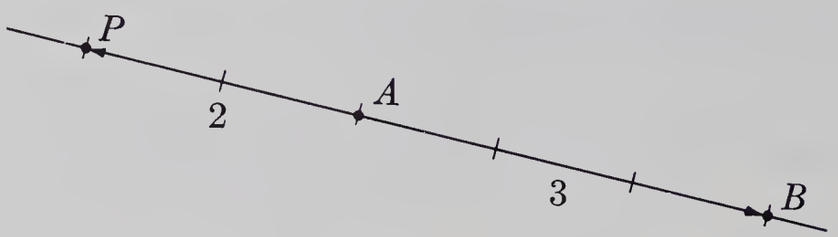
If it is understood that the domain of k is Re, we may refer to $\overrightarrow{AP} = k\overrightarrow{AB}$ as a vector representation or equation for the line l determined by the points A and B.

Example 1. Determine k so that P is the midpoint of line segment AB.

Solution: $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$ if P is the midpoint of AB; therefore $k = \frac{1}{2}$.

Example 2. If $\overrightarrow{AP} = -\frac{2}{3}\overrightarrow{AB}$, locate P relative to A and B.

Solution: $k < 0$ and so A lies between P and B; also $3|\overrightarrow{AP}| = 2|\overrightarrow{AB}|$, so that if $|\overrightarrow{AB}|$ is three units, then $|\overrightarrow{AP}|$ is two units.



The vector equation $\overrightarrow{AP} = k\overrightarrow{AB}$, $k \in Re$, is the equation of the line l determined by two distinct points A and B without regard to a standard co-ordinate system. The vectors involved all lie along the line l and are expressed relative to the two fixed points A and B. We proceed to locate the line l with respect to the frame of reference of a standard co-ordinate system and to find an equation for l with respect to this frame of reference.

Our first step is to consider a frame of reference consisting of a single point O. With respect to such a fixed point, any point P on the line l will have a position vector $\overrightarrow{OP}$ represented by the line segment from O to P; similarly, the fixed points A and B, which determine the line l, will have position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively (Figure 4.2).

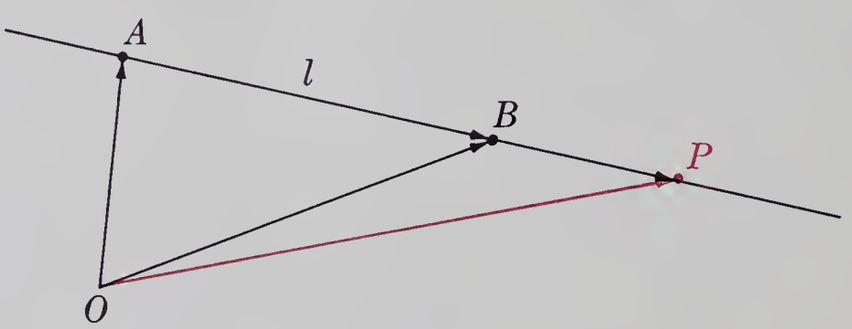


Figure 4.2

Now

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + k\overrightarrow{AB}, \qquad k \in Re,\\ &= \overrightarrow{OA} + k(\overrightarrow{OB} - \overrightarrow{OA}), \qquad \text{since } \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}\\ &= (1-k)\overrightarrow{OA} + k\overrightarrow{OB}.\end{aligned}$$

Then

$$\overrightarrow{OP} = (1 - k)\overrightarrow{OA} + k\overrightarrow{OB}, \qquad k \in Re,$$

is called the positional vector equation of line l in terms of the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ of the points A and B determining l. If we denote vector $\overrightarrow{OP}$ by $\mathbf{r}$, $\overrightarrow{OA}$ by $\mathbf{r}_1$, and $\overrightarrow{OB}$ by $\mathbf{r}_2$, this equation becomes

$$\mathbf{r} = (1 - k)\,\mathbf{r}_1 + k\,\mathbf{r}_2.$$

A slightly different form of the vector equation of line l is developed in question (7), Exercise 4.1. This second form will be used in some of the later developments.

By restricting k to subsets of Re, we get vector equations for various parts of line l.

Domain of k	$\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2$ represents
0	P at A
1	P at B
$0 \leq k \leq 1$	line segment AB
$k \geq 0$	ray AB
$k \leq 1$	ray BA
Re	line l

Note that if A and B have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, then

$$\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2$$

is the position vector of point P such that $\overrightarrow{AP} = k\overrightarrow{AB}$.

Example 3. Interpret the equation $\mathbf{r} = \frac{2}{3}\mathbf{r}_1 + \frac{1}{3}\mathbf{r}_2$.

Solution: $\mathbf{r}$ is the position vector of a point P on the line AB such that $\overrightarrow{AP} = k\overrightarrow{AB} = \frac{1}{3}\overrightarrow{AB}$; thus the equation represents a point P that divides line segment AB in the ratio 1:2.

Example 4. What figure does the equation $\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2$, $0 \leq k \leq \frac{1}{2}$, represent?

Solution: This equation represents the line segment joining A to the midpoint of line segment AB.

In our development thus far, we have placed no restrictions on the location of the point O and the line l; they may be in 2-space or in 3-space and the vector equation

$$\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2$$

will be the same in either situation. In either case, the point O and the line l determine a plane if O is not on l.

In succeeding sections we shall interpret the vector equation of a line relative to a Cartesian co-ordinate system first in the plane and then in 3-space.

EXERCISE 4.1

1. If $\overrightarrow{AP} = k\overrightarrow{AB}$, determine P relative to points A and B for the following real values of k (include a diagram).

   (a) $\frac{1}{3}$ (b) 2 (c) -1 (d) $\frac{5}{4}$
   (e) $-\frac{3}{2}$ (f) $\sqrt{2}$ (g) $0 \leq k \leq 1$ (h) $k > \frac{1}{2}$
   (i) $k = \{0, 1, 2\}$ (j) $2k < 1$ (k) $k < 0$ or $k > 1$ (l) $k \in \mathrm{I}$

2. Determine k if $\overrightarrow{AP} = k\overrightarrow{AB}$ in the following situations.
   (a) P divides line segment AB in the ratio 2:3.
   (b) P is twice as far from A as from B.
   (c) P is twice as far from B as from A.

3. If $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are position vectors of P, A, and B, relative to a point O, interpret the following equations (include a diagram).

   (a) $\mathbf{r} = \frac{1}{2}\mathbf{r}_1 + \frac{1}{2}\mathbf{r}_2$ (b) $\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{2}{3}\mathbf{r}_2$
   (c) $\mathbf{r} = \frac{1}{5}\mathbf{r}_1 + \frac{4}{5}\mathbf{r}_2$ (d) $\mathbf{r} = -\mathbf{r}_1 + 2\mathbf{r}_2$
   (e) $\mathbf{r} = -\frac{1}{2}\mathbf{r}_1 + \frac{3}{2}\mathbf{r}_2$ (f) $\mathbf{r} = 3\mathbf{r}_1 - 2\mathbf{r}_2$

4. Interpret the equation $\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2$ for the following sets of real values for k.

   (a) $\frac{1}{3} \leq k \leq \frac{2}{3}$ (b) $\frac{2}{3} \leq k \leq 1$
   (c) $-1 \leq k \leq 0$ (d) $1 \leq k \leq 2$
   (e) $-1 \leq k \leq \frac{1}{2}$ (f) $-\frac{1}{2} \leq k \leq \frac{3}{2}$

5. Prove that if points A and B have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, then point P which divides line segment AB in the ratio $p : q$ $(p > 0,\ q > 0)$ has position vector

$$\mathbf{r} = \frac{q\mathbf{r}_1 + p\mathbf{r}_2}{p + q}.$$

6. Interpret the result in question (5) in the case that p or q or both are negative numbers.

7. A line l is determined by a point A on the line and a vector $\mathbf{s}$ collinear with the line. Show that the line has vector equation

$$\mathbf{r} = \mathbf{r}_1 + k\mathbf{s},$$

   where $\mathbf{r}_1$ and $\mathbf{r}$ are position vectors, with respect to a fixed point O, of point A and an arbitrary point P on l.

8. Rewrite the equations of the points in question (3) in the form $\mathbf{r} = \mathbf{r}_1 + k\mathbf{s}$, where $\mathbf{s} = \mathbf{r}_2 - \mathbf{r}_1$.

4.2. Parametric Equations of a Line in Two-Space

We have developed the positional vector equation

$$\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2 , \quad k \in Re ,$$

for the line determined by two points A and B. $\mathbf{r}_1$ and $\mathbf{r}_2$ are position vectors, with respect to a fixed point O, of the fixed points A and B, and $\mathbf{r}$ is the position vector of an arbitrary point P on the line, the position of P on the line depending on the real value assigned to k.

We now introduce a Cartesian co-ordinate system in the plane of O and the line l by selecting two mutually perpendicular lines through point O and labelling them as the x- and y-axes in the usual manner (Figure 4.3).

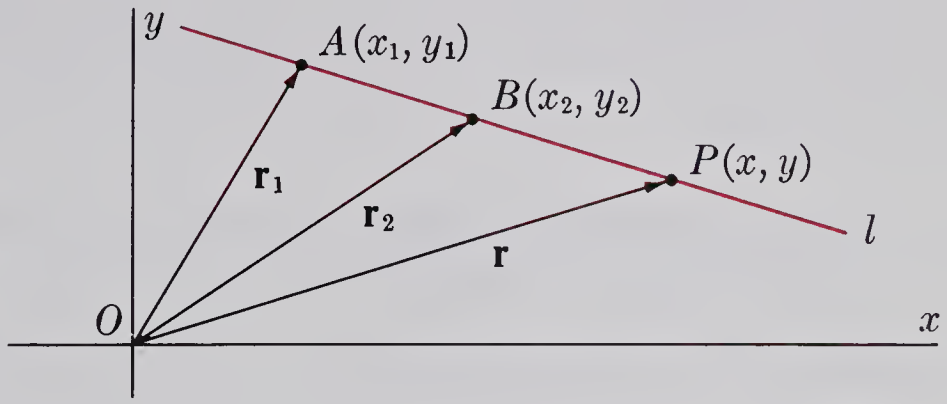


Figure 4.3

The point A has co-ordinates (x_1, y_1) relative to this co-ordinate system and the position vector $\mathbf{r}_1$ is the algebraic vector (x_1, y_1); similarly, B has co-ordinates (x_2, y_2) and P has co-ordinates (x, y). Thus

$$\mathbf{r}_1 = (x_1, y_1) , \quad \mathbf{r}_2 = (x_2, y_2) , \quad \mathbf{r} = (x, y) ,$$

and the vector equation for line l can be written

$$(x, y) = (1 - k)(x_1, y_1) + k(x_2, y_2) .$$

Example 1. Find the vector equation of the line l determined by the points $A(4, 3)$ and $B(-2, 9)$.

Solution: The vector equation of l is

$$(x, y) = (1 - k)(4, 3) + k(-2, 9) .$$

Now the vector equation

$$(x, y) = (1 - k)(x_1, y_1) + k(x_2, y_2)$$

can be written

$$\begin{aligned}(x, y) &= [(1 - k)x_1, \ (1 - k)y_1] + (kx_2, ky_2) \\ &= [(1 - k)x_1 + kx_2 , \ (1 - k)y_1 + ky_2]\end{aligned}$$

by our rules for multiplication of algebraic vectors by a scalar and for addition of algebraic vectors. Identifying components in these two vectors (2-dimensional vectors are equal if and only if corresponding components are equal) yields the two equations

$$\left.\begin{aligned} x &= (1-k)x_1 + kx_2 \\ y &= (1-k)y_1 + ky_2 \end{aligned}\right\} \quad k \in Re\,.$$

These are the parametric equations of the line determined by the points $A(x_1, y_1)$ and $B(x_2, y_2)$; the equations are given in terms of the parameter k. Note that a line in 2-space has a *pair* of parametric equations.

Also note that every point on the line determined by the points $A(x_1, y_1)$ and $B(x_2, y_2)$ has co-ordinates $(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$, where $\alpha + \beta = 1$.

Another useful form for the parametric equations of the line l is developed in question (7), Exercise 4.2.

Example 2. Find the parametric equations of the line l determined by the points $A(4, 3)$ and $B(-2, 9)$ (see Example 1).

Solution: The parametric equations of l are

$$\begin{aligned} x &= (1-k)(4) + k(-2) = 4 - 6k, \\ y &= (1-k)(3) + k(9) = 3 + 6k\,. \end{aligned}$$

We may find these by direct substitution in the parametric equations

$$\begin{aligned} x &= (1-k)x_1 + kx_2, \\ y &= (1-k)y_1 + ky_2, \end{aligned}$$

or by writing the vector equation of the required line in the form

$$(x, y) = (1-k)(4, 3) + k(-2, 9)$$

as in Example 1, and then combining on the right side

$$(x, y) = [(1-k)(4) + k(-2),\ (1-k)(3) + k(9)],$$

and equating corresponding components.

Of course, any restriction of k to a subset of Re in the parametric equations

$$\begin{aligned} x &= (1-k)x_1 + kx_2 \\ y &= (1-k)y_1 + ky_2 \end{aligned}$$

yields the parametric equations of a part of the line (a point, a set of points, a line segment, a ray, etc.) determined by $A(x_1, y_1)$ and $B(x_2, y_2)$.

Example 3. Find the midpoint of the line segment AB of the preceding examples.

Solution: The parametric equations of line AB are

$$x = 4 - 6k, \quad y = 3 + 6k .$$

The midpoint of the line segment AB is the point corresponding to the value $\frac{1}{2}$ of k; its co-ordinates are then

$$4 - 6(\tfrac{1}{2}) \text{ and } 3 + 6(\tfrac{1}{2}) .$$

The required point has co-ordinates $(1, 6)$.

Example 4. Check that point $P(1, 6)$ is the midpoint of line segment AB determined by $A(4, 3)$ and $B(-2, 9)$.

Solution: We need only show that $\overrightarrow{AP} = \overrightarrow{PB}$.

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (1, 6) - (4, 3) = (-3, 3) .$$
$$\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = (-2, 9) - (1, 6) = (-3, 3) .$$

Therefore, P is the midpoint of line segment AB.

Example 5. Locate the point with co-ordinates $(-2x_1 + 3x_2, \ -2y_1 + 3y_2)$.

Solution: The equations

$$x = (1 - 3)x_1 + 3x_2$$
$$y = (1 - 3)y_1 + 3y_2$$

are the equations of the part of the line l determined by $A(x_1, y_1)$ and $B(x_2, y_2)$ corresponding to the single value 3 of k. This is the single point C such that $\overrightarrow{AC} = 3\,\overrightarrow{AB}$. Thus P is on l on the side of B opposite to that of A and a distance from B that is twice that of B from A (Figure 4.4).

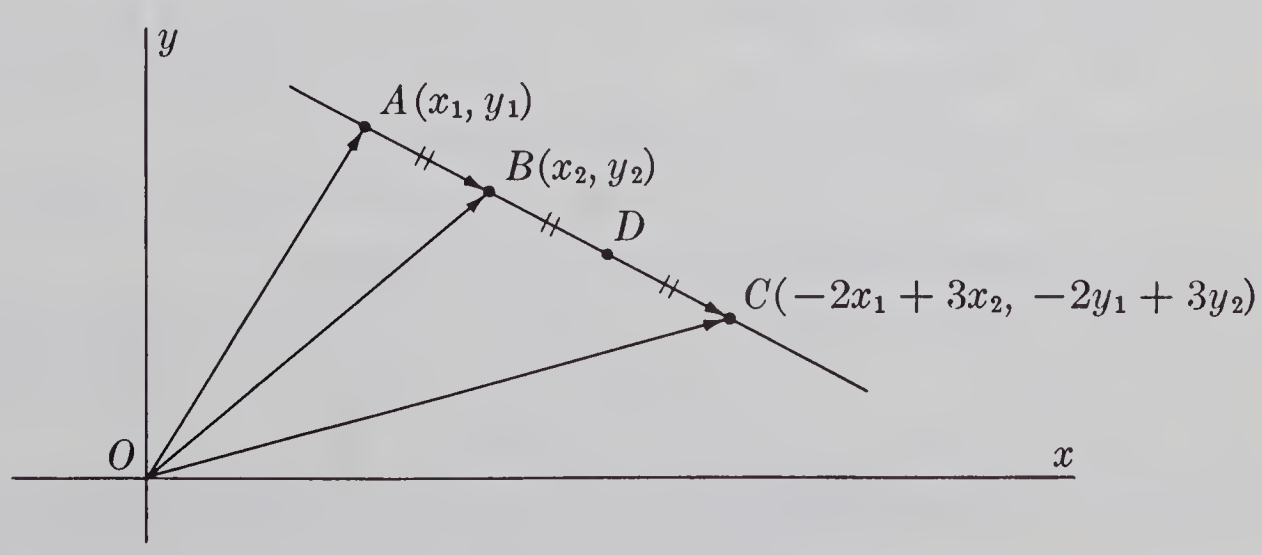


Figure 4.4

Example 6. Determine the figure with parametric equations

$$\left.\begin{aligned} x &= (1-k)x_1 + kx_2 \\ y &= (1-k)y_1 + ky_2 \end{aligned}\right\} \quad 1 \leq k \leq 3,\ k \in Re.$$

Solution: These are the parametric equations of the line segment BC shown in Figure 4.4.

Note that if k had been restricted to I in this example, with $1 \leq k \leq 3$, the parametric equations would have determined only the three points B, D, and C.

EXERCISE 4.2

1. Give vector equations and parametric equations (with respect to a Cartesian co-ordinate system) for lines determined by the pairs of points with the given co-ordinates (include a diagram).
   (a) $(1, 1)$ and $(5, 2)$ (b) $(3, 7)$ and $(-4, 2)$
   (c) $(6, 6)$ and $(0, 0)$ (d) $(-1, 0)$ and $(0, 7)$
   (e) $(3, -3)$ and $(0, 5)$ (f) $(-2, 3)$ and $(-4, 7)$
   (g) $(-2, 4)$ and $(-5, -6)$ (h) $(-3, -1)$ and $(-7, -2)$
   (i) $(4, \frac{1}{3})$ and $(-\frac{1}{2}, 0)$ (j) $(5, \sqrt{2})$ and $(\sqrt{3}, 6)$
2. Find the midpoints of the line segments determined by the pairs of points in question (1).
3. Find the co-ordinates of the point P dividing line segment AB, determined by $A(-5, 2)$ and $B(9, 5)$, in the following ratios.
   (a) $1:2$ (b) $3:5$ (c) $-1:2$ (d) $2:-1$
4. Locate the following points with respect to the points $A(x_1, y_1)$ and $B(x_2, y_2)$ (include a diagram).
   (a) $(\frac{1}{3}x_1 + \frac{2}{3}x_2, \frac{1}{3}y_1 + \frac{2}{3}y_2)$ (b) $(2x_1 - x_2, 2y_1 - y_2)$
   (c) $(\frac{4}{3}x_1 - \frac{1}{3}x_2, \frac{4}{3}y_1 - \frac{1}{3}y_2)$ (d) $(-4x_1 + 5x_2, -4y_1 + 5y_2)$.
5. Determine the figure with parametric equations
   $$x = (1-k)(4) + k(-3), \qquad y = (1-k)(-5) + k(6)$$
   for the following sets of values for k (use a diagram).
   (a) Re (b) 0 (c) 1 (d) $\frac{1}{2}$ (e) 2 (f) $\{0, 1, 2\}$
   (g) $\{-1, 0, 1\}$ (h) $\{-1 \leq k \leq 1,\ k \in Re\}$ (i) $\{k \geq 0,\ k \in Re\}$
   (j) $\{k \leq 0,\ k \in Re\}$ (k) $\{k \geq 1,\ k \in Re\} \cup \{k \leq 0,\ k \in Re\}$.
6. Determine the co-ordinates of two points on the line with parametric equations
   $$x = 5 - 3t,\ y = 4 + 7t.$$
   Is the point $(14, -17)$ on this line? Is the point $(-\frac{1}{2}, \frac{27}{2})$?

7. Show that the vector equation $\mathbf{r} = \mathbf{r}_1 + k\mathbf{s}$ of line l developed in question (7), Exercise 4.1 gives rise to the parametric equations

$$x = x_1 + ka, \qquad y = y_1 + kb, \qquad k \in Re,$$

if point A on l has co-ordinates (x_1, y_1) and vector $\mathbf{s}$, collinear with l, is $\mathbf{s} = (a, b)$ with respect to the chosen co-ordinate system.

8. Write vector and parametric equations of the form given in question (7) for line l if A has co-ordinates $(2, 3)$ and $\mathbf{s}$ is the given vector.

(a) $\mathbf{s} = (3, 5)$ (b) $\mathbf{s} = (-2, 1)$ (c) $\mathbf{s} = (0, -5)$ (d) $\mathbf{s} = (-4, 0)$

(e) $\mathbf{s} = \overrightarrow{AB}$ with $B(4, 9)$ (f) $\mathbf{s} = \overrightarrow{AB}$ with $B = (-5, -1)$

9. State the co-ordinates of the points corresponding to parameter values 0 and 1 on the lines with the following equations. State also the vector (a, b) collinear with each line.

(a) $x = 3 + 2k,\ y = 4 - 3k$ (b) $x = -1 - 2k,\ y = 3 - k$

(c) $x = 4k,\ y = 1 - 2k$ (d) $x = 3,\ y = -2 + 5k$

(e) $x = k,\ y = -2k$ (f) $x = 4 - k,\ y = -1$

10. Describe the location of the line l with parametric equations

$$x = 4 - 2t,\ y = 3\,.$$

11. Describe the location of the line with parametric equations

$$x = -3,\ y = 5 + 3t\,.$$

12. Find the parametric equations of the line through the origin and the point (x_1, y_1).

13. Find the parametric equations of the lines through point $A(x_1, y_1)$ and

(a) parallel to the x-axis, (b) parallel to the y-axis.

14. The vector equation of line l is

$$(x, y) = (1 - k)(1, 2) + k(-2, 3)\,.$$

On this line, points A, B, C, and D correspond to the parametric values 0, 1, 2, and 3, respectively. Show that

$$\overrightarrow{AC} = 2\overrightarrow{AB}, \quad \overrightarrow{AD} = 3\overrightarrow{AB}, \quad \overrightarrow{AC} = \tfrac{2}{3}\overrightarrow{AD}\,.$$

15. If the vector equation of line l is

$$(x, y) = (1 - k)(x_1, y_1) + k(x_2, y_2),$$

and if points A, B, and P correspond to the values 0, 1, and t of the parameter k, respectively, prove that

$$\overrightarrow{AP} = t\overrightarrow{AB}.$$

16. If points R, S, and T correspond to the parametric values 0, t_1, and t_2, respectively, for the line in question (15), prove that

$$\overrightarrow{RT} = \frac{t_2}{t_1}\overrightarrow{RS}\,, \qquad \text{if} \qquad t_1 \neq 0\,.$$

4.3. The Linear Equation of a Line in Two-Space

A line l in 2-space determines a unique line l' through the origin and perpendicular to l (Figure 4.5); l' is called the normal axis of l. The direction of the normal axis l' (and hence of l) is known when a point $M(a, b)$, different from the origin, is known to lie on the normal axis. In this case, l' is collinear with vector $\overrightarrow{OM} = (a, b)$.

A particular line l in 2-space is thus determined by

(i) a point $P_1(x_1, y_1)$ on l

and

(ii) a point $M(a, b)$ on the normal axis l'.

We obtain an equation for l by noting that, if $P(x, y)$ is any point on l different from P_1, then the vector $\overrightarrow{P_1P}$ is perpendicular to vector $\overrightarrow{OM}$.

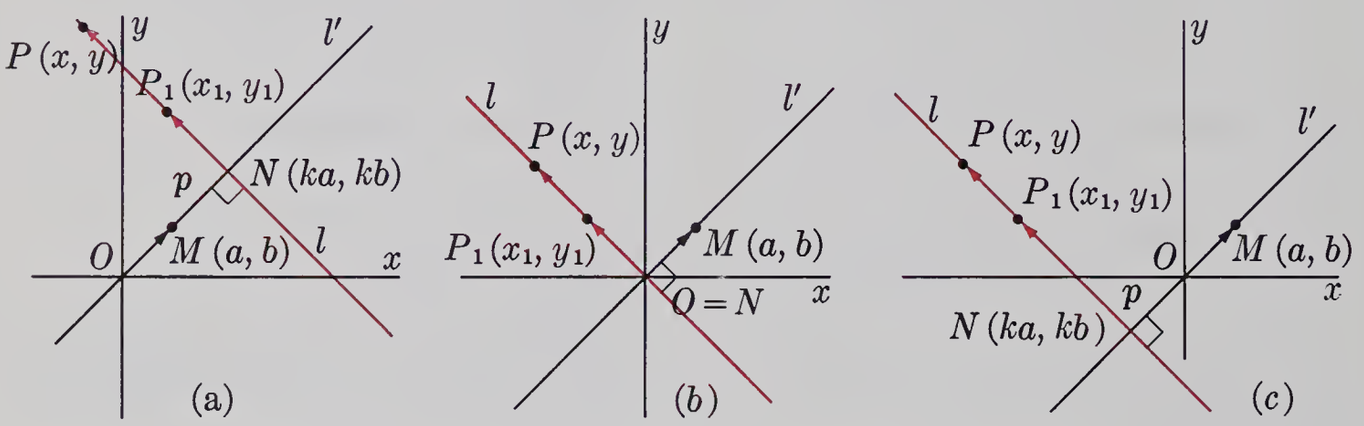


Figure 4.5

Theorem. Let line l in 2-space be determined by

(i) point $P_1(x_1, y_1)$ on l,

(ii) point $M(a, b)$ on l', the normal axis of l.

Then l has the equation

$$ax + by = ax_1 + by_1 .$$

Proof: Let $P(x, y)$ be any point on l different from P_1. Then

$$\overrightarrow{P_1P} = (x - x_1, y - y_1)$$

is perpendicular to vector $\overrightarrow{OM} = (a, b)$. But $\overrightarrow{OM} \perp \overrightarrow{P_1P}$ if and only if

$$(a, b) \cdot (x - x_1, y - y_1) = 0 ;$$

that is,

$$ax - ax_1 + by - by_1 = 0 ,$$

or

$$ax + by = ax_1 + by_1 .$$

Thus the co-ordinates of every point P on l satisfy this equation, which is therefore the equation of line l.

This equation is the one that has been studied earlier, the linear equation

$$ax + by = c.$$

Example 1. Find the equation of line l if l passes through $P_1(5, 7)$ and its normal axis l' lies on $M(-3, 1)$.

Solution:

First Method (from first principles): Let $P(x, y)$ be any point on l distinct from P_1. Then $\overrightarrow{OM} = (-3, 1)$, $\overrightarrow{P_1P} = (x - 5, y - 7)$, and $\overrightarrow{OM} \perp \overrightarrow{P_1P}$. Therefore,

$$(-3, 1) \cdot (x - 5, y - 7) = 0$$

and hence

$$-3x + y = -8$$

is the required equation.

Second Method: Since $\overrightarrow{OM} = (-3, 1)$ is collinear with the normal axis, the equation of l is of the form

$$-3x + y = c\,,$$

where

$$\begin{aligned} c &= (-3)(5) + 1(7) \\ &= -8\,, \end{aligned}$$

since $P_1(5, 7)$ lies on l. This gives us the same equation as obtained by the first method.

It is sometimes useful to take the point of intersection N of line l and its normal axis l' as the point P_1 on l (Figure 4.5). Point N will have co-ordinates (ka, kb) for some real number k, since $\overrightarrow{ON} = k\overrightarrow{OM}$. Then, if $P(x, y)$ is any different point on l, we have $\overrightarrow{NP} \perp \overrightarrow{OM}$ so that

$$(a, b) \cdot (x - ka, y - kb) = 0\,,$$

or

$$ax + by = k(a^2 + b^2)\,.$$

Example 2. Find the point of intersection of the line l with equation $2x - 5y = 12$ and its normal axis l'.

Solution: We note first that l' passes through point $M(2, -5)$, and secondly, that the point of intersection N of l and l' has co-ordinates $(2k, -5k)$, where k is such that

$$\begin{aligned} 2x - 5y &= k(2^2 + (-5)^2) \\ &= 12\,. \\ 29k &= 12 \text{ and } k = \tfrac{12}{29}\,. \end{aligned}$$

The point of intersection is $N(\tfrac{24}{29}, -\tfrac{60}{29})$.

When we know the co-ordinates (ka, kb) of the point of intersection N of l and l', we can immediately find the perpendicular distance p from the origin to line l. This distance can be considered as

(i) the length of vector $\overrightarrow{ON} = (ka, kb)$,

or

(ii) the length of line segment ON.

The required distance is $|k|\sqrt{a^2 + b^2}$.

Example 3. Find the distance p from the origin to the line l with equation $2x - 5y = 12$ (see Example 2).

Solution: We found that $k = \frac{12}{29}$; the required distance p is

$$\tfrac{12}{29}\sqrt{2^2 + (-5)^2} = \tfrac{12}{29}\sqrt{29}\,.$$

The general linear equation

$$ax + by = c, \qquad a, b, c \in Re$$

may be rewritten in the form

$$ax + by = \left(\frac{c}{a^2 + b^2}\right)(a^2 + b^2)\,,$$

so that

$$ax + by = k(a^2 + b^2), \qquad \text{where } k = \frac{c}{a^2 + b^2}\,.$$

Note that, when the equation is written in this form,

(i) point $M(a, b)$ lies on the normal axis l' of l (why?);

(ii) if l and l' intersect in point N, then $\overrightarrow{ON} = k\overrightarrow{OM}$ and so N has co-ordinates (ka, kb);

(iii) the (perpendicular) distance p from the origin to line l is

$$p = |\overrightarrow{ON}| = |k|\sqrt{a^2 + b^2}\,.$$

Example 4. Find the equations of the lines perpendicular to the vector $(4, 1)$ and 7 units distance from the origin (two answers).

Solution: The equations of the lines are of the form

$$4x + y = c\,.$$

If we write this equation as

$$4x + y = \frac{c}{\sqrt{17}}(17)\,,$$

then the distance from the origin to the line is $\frac{|c|}{\sqrt{17}}$. Therefore,

$$\frac{|c|}{\sqrt{17}} = 7$$

and so $c = 7\sqrt{17}$ or $-7\sqrt{17}$. The required equations are

$$4x + y = 7\sqrt{17} \quad \text{and} \quad 4x + y = -7\sqrt{17}\,.$$

Example 5. If line l has equation $5x + 3y = 7$, find

(a) a vector perpendicular to the direction of l,
(b) a point on the normal axis l',
(c) the point of intersection N of l and l',
(d) the distance from O to l.

Solution:

(a) Vector $(5, 3)$ is perpendicular to the direction of l.
(b) Point $M(5, 3)$ lies on l'.
(c) Writing $5x + 3y = \frac{7}{34}(34)$, where $34 = 5^2 + 3^2$, we see that the point of intersection N of l and l' is such that $\overrightarrow{ON} = \frac{7}{34}\overrightarrow{OM}$; thus N has co-ordinates $(\frac{35}{34}, \frac{21}{34})$.
(d) The distance from O to l is $\frac{7}{34}\sqrt{34}$ or $\frac{7}{\sqrt{34}}$. This may be checked by using the distance formula to find the length of line segment ON.

Example 6. Find the distance from the point $Q(7, 4)$ to the line l whose equation is $3x + 5y = 19$.

Solution: The point $M(3, 5)$ lies on the normal axis l' and the point $P_1(3, 2)$ lies on l, since $3(3) + 5(2) = 19$. Thus we may sketch the location of l (Figure 4.6.)

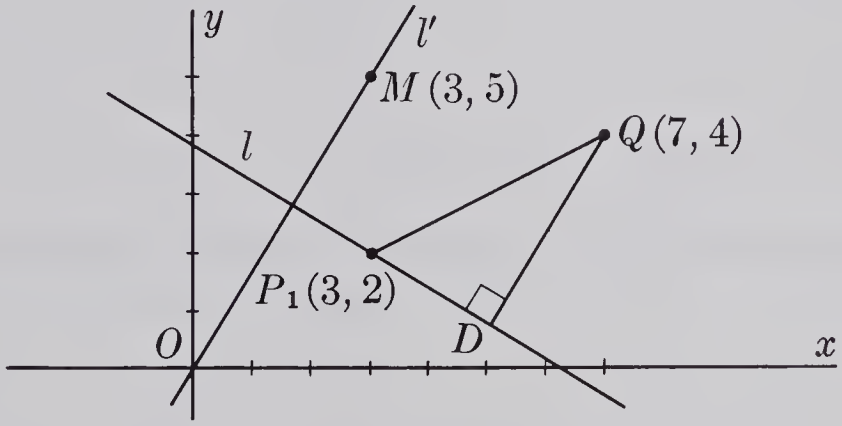


Figure 4.6

The required distance QD is the projection of vector $\overrightarrow{P_1Q}$ on a vector perpendicular to the direction of l; we may take $\overrightarrow{OM} = (3, 5)$ as this vector.

$$\overrightarrow{P_1Q} = (7 - 3, 4 - 2) = (4, 2)\,.$$

Therefore,
$$\text{required distance} = \frac{|\overrightarrow{P_1Q} \cdot \overrightarrow{OM}|}{|\overrightarrow{OM}|}$$
$$= \frac{|(4, 2) \cdot (3, 5)|}{\sqrt{34}}$$
$$= \frac{22}{\sqrt{34}} = \frac{11}{17}\sqrt{34}\,.$$

Note that we could have used the point of intersection N of l and l' (or any other point on l) instead of P_1.

EXERCISE 4.3

1. Find the equation of line l through point P_1 if its normal axis lies on point M; use two methods in (a)-(f). Locate the line in the plane in each case.
   (a) $M(1,1)$, $P_1(5,5)$ (b) $M(3,3)$, $P_1(-1,0)$
   (c) $M(-1,4)$, $P_1(2,-8)$ (d) $M(-1,2)$, $P_1(4,3)$
   (e) $M(-3,1)$, $P_1(9,-3)$ (f) $M(-2,-3)$, $P_1(-4,7)$
   (g) $M(3,0)$, $P_1(10,0)$ (h) $M(0,4)$, $P_1(0,9)$
   (i) $M(\frac{1}{2},\frac{1}{3})$, $P_1(3,2)$ (j) $M(-2,-5)$, $P_1(4,-1)$
   (k) $M(3,7)$, $P_1(0,0)$ (l) $M(-2,5)$, $P_1(0,5)$

2. Find the equation of l if l and l' intersect in the following points N.
   (a) $N(3,5)$ (b) $N(-3,7)$ (c) $N(-2,-7)$ (d) $N(0,4)$ (e) $N(0,0)$

3. Find the equations of the lines perpendicular to the given vector and at the given distance from the origin.
   (a) $(3,4)$, 5 (b) $(-2,7)$, 3 (c) $(4,-6)$, 10

4. Find the intersection of line l and its normal axis l', and sketch the lines l with the following equations.
   (a) $3x + 4y = 13$ (b) $3x = 7$
   (c) $2x - 5y = 9$ (d) $-3x + 7y = 3$
   (e) $2y = -5$ (f) $4x - 6y = -3$
   (g) $-3x + 2y = 13$ (h) $-x - 5y = -3$

5. Find the distance from the origin to line l in question (4).

6. Find the distance from point Q to the line whose equation is given.
   (a) $Q(3,-7)$, $2x - y = 5$ (b) $Q(-3,-1)$, $x + 5y = -8$
   (c) $Q(-1,6)$, $5x - 3y = -2$ (d) $Q(2,5)$, $-2x + 7y = 3$

7. Prove that the distance from the origin to the line with equation $ax + by = c$ is
$$\frac{|c|}{\sqrt{a^2 + b^2}}.$$

8. Prove that the distance from the point $P_1(x_1, y_1)$ to the line with equation $ax + by = c$ is
$$\frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.$$

9. Prove that the origin lies between the points $A(a,b)$ and $B(ka,kb)$ whenever $k < 0$.

10. Show that the equation of line l can be written in the form
$$x \cos\omega + y \sin\omega = p$$
where ω is the angle that the normal axis l' makes with the positive x-axis ($0° \leq \omega \leq 360°$) and p is the distance from O to l.

4.4. Direction Numbers · Symmetric Equations of a Line in Two-Space

We have seen that a line l in the plane is determined by two points P_1 and P_2 or by a point P_1 and the direction of the normal axis l'. The line l is also determined by a point $P_1(x_1, y_1)$ and the direction of the line l itself, given by the angle α that l makes with the positive x-axis $(0° \leq \alpha \leq 180°)$.

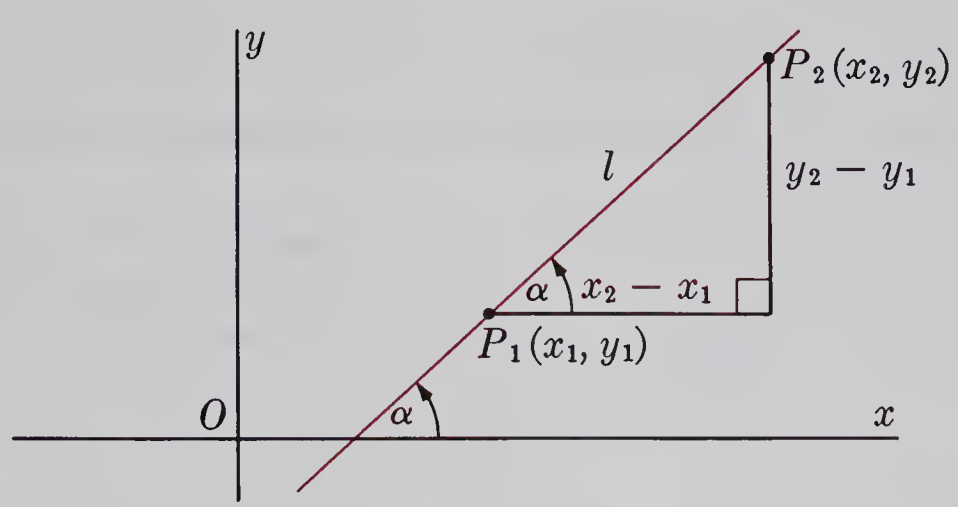


Figure 4.7

Now suppose (Figure 4.7) that we were able to stand at the point $P_1(x_1, y_1)$ on line l. If we walked a directed distance $x_2 - x_1$ units parallel to the x-axis, and then $y_2 - y_1$ units parallel to the y-axis, we would reach the point $P_2(x_2, y_2)$ on line l. In this sense, the ordered pair of real numbers $(x_2 - x_1, y_2 - y_1)$ determines the *direction* of the line l. For this reason, such an ordered pair of real numbers is called a set of direction numbers for the line l.

Every ordered pair of real numbers (a, b) will thus determine the direction of an infinite family of parallel lines (Figure 4.8).

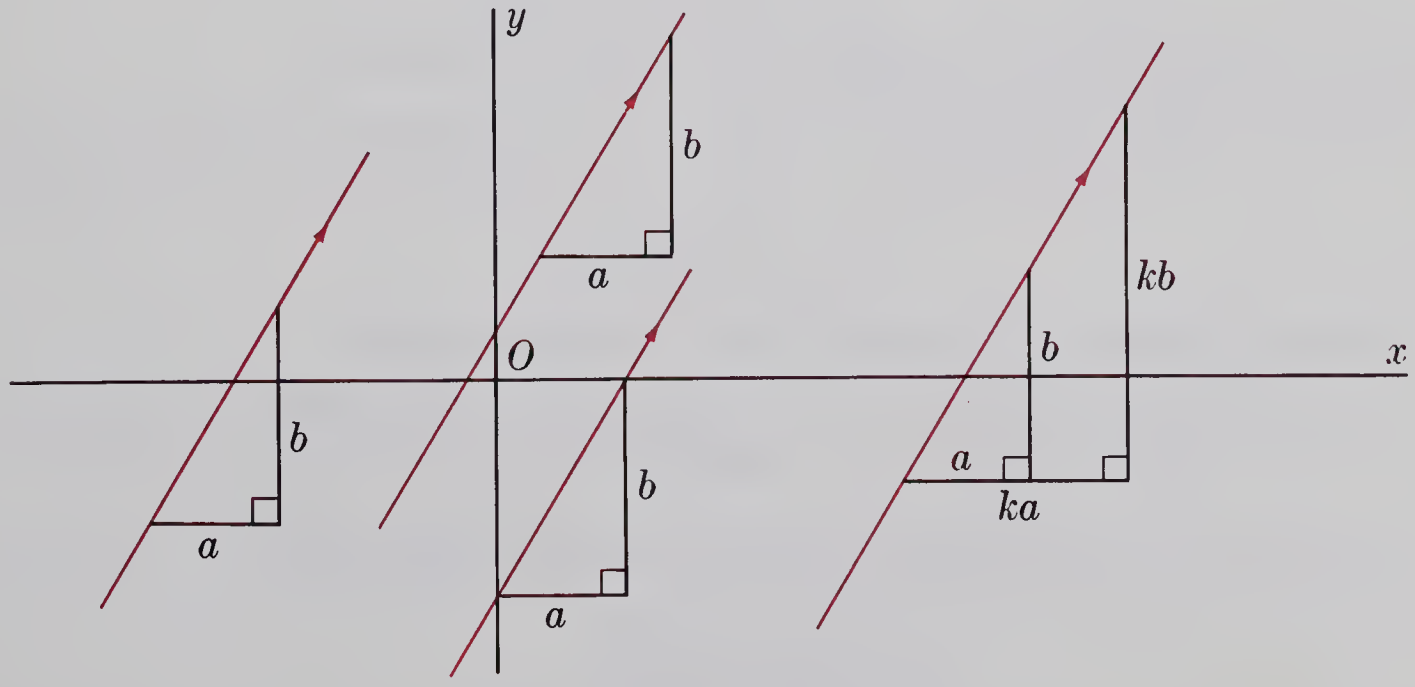


Figure 4.8

An application of the properties of similar triangles shows that any ordered pair (ka, kb), $k \neq 0$, determines the same infinite family of parallel lines. Thus if l has direction numbers $(3, 5)$, then l will also have direction numbers $(6, 10)$, $(9, 15)$, $(-3, -5)$, and, in general, $(3k, 5k)$.

Example 1. Find a general expression for the direction numbers of the line l through points $P_1(2, -5)$ and $P_2(3, 7)$.

Solution: Since $(x_2 - x_1, y_2 - y_1)$ is a set of direction numbers, then $(3 - 2, 7+5)$, or $(1, 12)$ is a set of direction numbers for l. This means that if we start at $P_1(2, -5)$ and move one unit parallel to the x-axis, then 12 units parallel to the y-axis, we will reach point P_2.

The general expression for the set of direction numbers of this line l is $(k, 12k)$, for $k \neq 0$. If we start at $P_1(2, -5)$ and move k units parallel to the x-axis, then $12k$ units parallel to the y-axis, we shall arrive at a point on line l.

We have now encountered another interpretation for an ordered pair of real numbers.

(1) (a, b) can be interpreted as the co-ordinates of a point A in the plane.

(2) (a, b) can be interpreted as the vector $\overrightarrow{OA}$.

(3) (a, b) can be interpreted as the direction of a family of parallel lines (having the direction of vector $\overrightarrow{OA}$), that is, as direction numbers for the family of parallel lines.

It should be clear from the context which interpretation is being used.

In Section 4.2 we determined the parametric equations

$$x = (1 - k)x_1 + kx_2$$
$$y = (1 - k)y_1 + ky_2$$

of the line l determined by two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

These equations may be rewritten as

$$x - x_1 = k(x_2 - x_1),$$
$$y - y_1 = k(y_2 - y_1),$$

and, by the elimination of k, combined into the single equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, \quad \text{provided } x_1 \neq x_2, \ y_1 \neq y_2.$$

This is a symmetric equation of the line l in 2-space. Of course,

$$\frac{x - x_2}{x_2 - x_1} = \frac{y - y_2}{y_2 - y_1}$$

is also a symmetric equation for this line.

Example 2. Find a symmetric equation of the line l through the points $P_1(3, 5)$ and $P_2(7, 10)$.

Solution: A symmetric equation of l is

$$\frac{x-3}{7-3} = \frac{y-5}{10-5} \quad \text{or} \quad \frac{x-7}{7-3} = \frac{y-10}{10-5};$$

that is,

$$\frac{x-3}{4} = \frac{y-5}{5} \quad \text{or} \quad \frac{x-7}{4} = \frac{y-10}{5}.$$

Setting each ratio in the left equation equal to the parameter k yields the parametric equations

$$x = 3 + 4k, \quad y = 5 + 5k.$$

(From the right equation we get $x = 7 + 4h$, $y = 10 + 5h$.) Finally, multiplying both members of the symmetric equation by 20 yields

$$5(x-3) = 4(y-5) \quad \text{or} \quad 5(x-7) = 4(y-10),$$

that is, $5x - 4y = -5$, the linear equation of l.

Example 3. Find a symmetric equation of the line l through the points $P_1(3, 5)$ and $P_2(7, 5)$.

Solution: Line l does not have a symmetric equation because $y_2 - y_1 = 0$. Parametric equations and a linear equation may be determined.

The symmetric equation of l may be modified by noting that the denominators form a set of direction numbers for l. Also, since

$$\frac{x-x_1}{k(x_2-x_1)} = \frac{y-y_1}{k(y_2-y_1)} \quad \text{for } k \neq 0,$$

any set of direction numbers may be used in the equation; that is, if l has direction numbers (a, b) and if l lies on $P_1(x_1, y_1)$, then l has equation

$$\frac{x-x_1}{a} = \frac{y-y_1}{b}.$$

Example 4. Find the linear equation of line l if it has direction parallel to vector $\mathbf{v} = (3, -2)$ and lies on the point $Q(2, 7)$.

Solution: The line l has direction numbers $(3, -2)$; its equation is

$$\frac{x-2}{3} = \frac{y-7}{-2}.$$

This equation may be written as $2x + 3y = 25$.

EXERCISE 4.4

1. Find (i) direction numbers (ii) a symmetric equation for the following lines.
   (a) line l through $P_1(3, 3)$ and $P_2(7, 9)$
   (b) line l through $P_1(-2, 5)$ and $P_2(8, 0)$
   (c) line l through $P_1(-3, -2)$ and $P_2(0, 7)$
   (d) line l through $P_1(-1, -2)$ and $P_2(-3, -4)$
   (e) line l with equation $3x + 7y = 10$
   (f) line l with equation $-2x + 9y = 18$
   (g) line l with equation $4x - 3y = 12$
   (h) line l with equations $x = 4 + 3t$, $y = 6 - 2t$
   (i) line l with equations $x = -3 - t$, $y = 4$
   (j) line l with equations $x = -2$, $y = 1 - 3t$
   (k) line l through $P_1(5, 2)$ if its normal axis lies on $A(3, -4)$
2. Find the equations of the lines l through $A(1, -5)$ with direction numbers
   (a) $(5, -1)$, (b) $(3, 7)$, (c) $(-4, -3)$, (d) $(5, 0)$.
3. Find the equation of line l through $P_1(-2, 3)$ if its normal axis has equation $3x - 5y = 7$.
4. What can be said about line l if its direction numbers are
   (a) $(k, 0)$, (b) $(0, k)$,
   for $k \in Re$, $k \neq 0$?

4.5. Direction Cosines and Angles for a Line in Two-Space

The symmetric equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, \qquad x_2 \neq x_1, \; y_2 \neq y_1,$$

may be modified as

$$\frac{\dfrac{x - x_1}{x_2 - x_1}}{d} = \frac{\dfrac{y - y_1}{y_2 - y_1}}{d}$$

with $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Now if α is the angle between line l and the positive x-axis $(0° \leq \alpha \leq 180°)$, and β is the angle between l and the positive y-axis $(-90° \leq \beta \leq 90°)$, we see from Figure 4.9 that

$$\cos \alpha = \frac{x_2 - x_1}{d} \quad \text{and} \quad \cos \beta = \frac{y_2 - y_1}{d}.$$

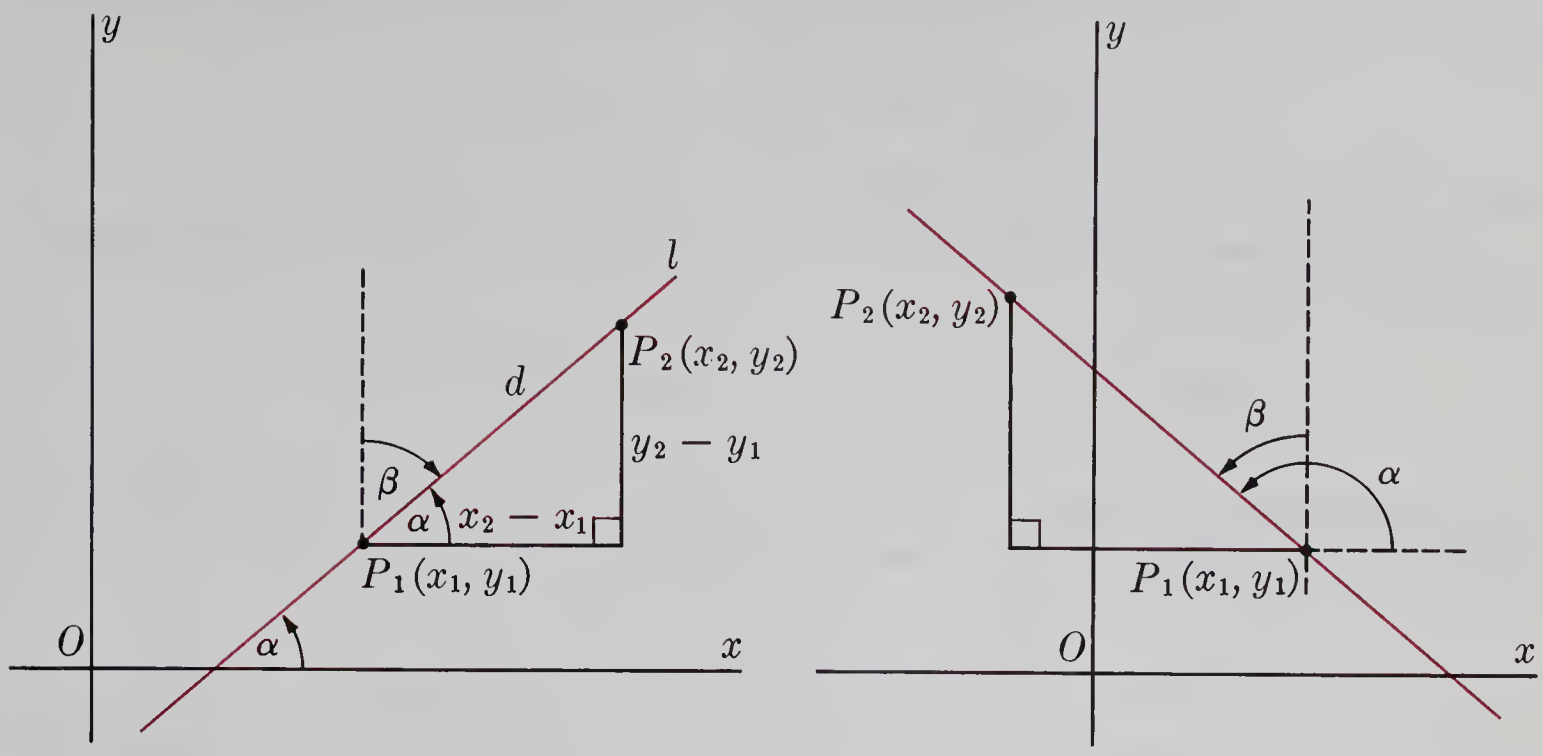


Figure 4.9

The equation of l can then be written in the form

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta}$$

α and β are called the direction angles of l, and $\cos\alpha$ and $\cos\beta$ are called the direction cosines of l.

Since $\cos\alpha$ and $\cos\beta$ can be either positive or negative, there are four possible patterns of signs for $(\cos\alpha, \cos\beta)$.

$$(+, +), \quad (-, +), \quad (-, -), \quad (+, -).$$

Now the line through point P with direction cosines $(-s, -t)$ has direction opposite to that of the line through P with direction cosines $(+s, +t)$. We do not distinguish between these lines. Nor do we distinguish between the lines through P with direction cosines $(-s, t)$ and $(s, -t)$.

The x-axis divides the plane into two half-planes, the upper half-plane, where y is positive, and the lower half-plane, where y is negative.

In order to specify a unique pair of direction angles for a line l, we shall adopt the following convention. We call that portion of l lying in the upper half-plane (assuming that l is not parallel to the x-axis) *the upper portion of* l. With the usual convention that angles measured in a counterclockwise direction are positive and angles measured in a clockwise direction are negative, we define α as the angle from the positive direction of the x-axis to the upper portion of l, and β as the angle from the upper portion of l to the positive direction of the y-axis. If l is parallel to the x-axis, $\alpha = 0°$ and $\beta = 90°$.

With this convention, $0 \leq \alpha < 180°$, $-90° < \beta \leq 90°$, $\alpha + \beta = 90°$, and *cos β is always positive.* It follows, from the equation in the preceding sentence, that

$$\cos^2\alpha + \cos^2\beta = \cos^2\alpha + \sin^2\alpha = 1 ;$$

that is, the sum of the squares of the direction cosines of l is unity.

Of course, the set of direction cosines $(\cos \alpha, \cos \beta)$ is also a set of direction numbers for l since

$$\cos \alpha = \frac{1}{d}(x_2 - x_1) \quad \text{and} \quad \cos \beta = \frac{1}{d}(y_2 - y_1) .$$

It is also possible to obtain the direction cosines of a line l from a set of direction numbers (Figure 4.10); if l has direction numbers (a, b), then

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos \beta = \frac{b}{\sqrt{a^2 + b^2}} .$$

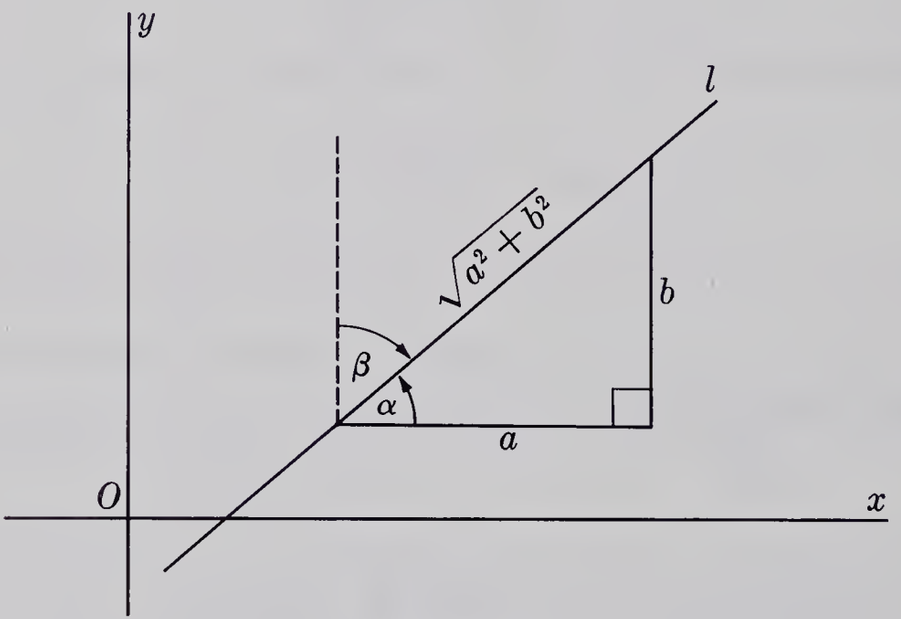


Figure 4.10

Example 1. Find the direction cosines and direction angles, to the nearest degree, of line l through points $P_1(5, -4)$ and $P_2(-1, 7)$ (Figure 4.11).

Solution: l has direction numbers $(-1 - 5, 7 + 4)$ or $(-6, 11)$. Therefore,

$$\cos \alpha = \frac{-6}{\sqrt{157}} \quad \text{and} \quad \cos \beta = \frac{11}{\sqrt{157}}, \quad \text{by our convention,}$$

$$\simeq -.479 . \qquad \simeq .878 .$$

$$\alpha = 119° . \qquad \beta = -29° .$$

$$\alpha + \beta = 90° .$$

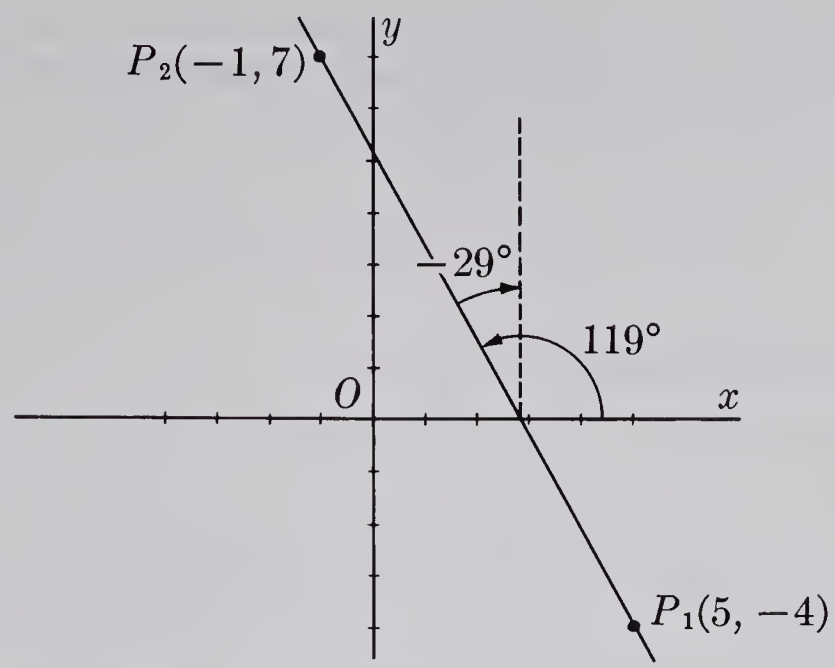


Figure 4.11

Example 2. Find the direction angles, to the nearest degree, of l if its direction cosines, without regard to our convention on signs, are

(a) $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, (b) $(-\frac{2}{3}, -\frac{\sqrt{5}}{3})$.

Solution:

(a) We take $\cos\alpha = -\frac{1}{2}$ and $\cos\beta = \frac{\sqrt{3}}{2}$ by our convention.

$\cos\alpha = -\frac{1}{2}$. $\alpha = 120°$.

$\cos\beta = \frac{\sqrt{3}}{2}$. $\beta = -30°$.

$\alpha + \beta = 90°$.

(b) We take $\cos\alpha = \frac{2}{3}$ and $\cos\beta = \frac{\sqrt{5}}{3}$ by our convention.

$\cos\alpha = 0.667$. $\alpha = 48°$.

$\cos\beta = 0.745$. $\beta = 42°$.

$\alpha + \beta = 90°$.

Since the normal axis l' of l is collinear with the vector (a, b) if l has equation $ax + by = c$, then l' has direction numbers (a, b). Similarly, l then has direction parallel to the vector $(-b, a)$, since

$$(a, b) \cdot (-b, a) = 0\,,$$

so that l has direction numbers $(-b, a)$.

Example 3. Find the direction cosines and a symmetric equation of the line l whose equation is $-2x + 5y = 10$.

Solution:

(a) The normal axis l' has direction numbers $(-2, 5)$; thus, the line l has direction numbers $(5, 2)$, since $(-2, 5) \cdot (5, 2) = 0$. Therefore,

$$\cos\alpha = \frac{5}{\sqrt{29}} \quad \text{and} \quad \cos\beta = \frac{2}{\sqrt{29}}\,.$$

(b) In order to write a symmetric equation of line l, we need the co-ordinates of a point on l. These are easily determined; for example, $(5, 0)$ and $(0, 2)$ are the co-ordinates of points on l. Then

$$\frac{x-5}{5} = \frac{y-0}{2} \quad \text{and} \quad \frac{x-0}{5} = \frac{y-2}{2}$$

are symmetric equations for l.

EXERCISE 4.5

1. Find (a) the direction cosines, (b) the direction angles of the lines in question (1), Exercise 4.4.
2. Find the equations of lines l through $A(2, 3)$ with direction angles
   (a) $45°, 45°$, (b) $120°, -30°$, (c) $165°, -75°$.
3. Find the direction angles, to the nearest degree, of the lines l whose direction cosines, without regard to our convention on signs, are given as
   (a) $\frac{3}{4}, \frac{\sqrt{7}}{4}$, (b) $-\frac{1}{3}, \frac{2\sqrt{2}}{3}$, (c) $\frac{3}{5}, -\frac{4}{5}$, (d) $-\frac{5}{13}, -\frac{12}{13}$.
4. Find the equation of line l if its distance from the origin is 5 units and its normal axis makes an angle of $30°$ with the positive x-axis.

4.6. Equations of the Line in Three-Space

The line l determined by two points A and B with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to a fixed point O, has vector equation

$$\mathbf{r} = (1-k)\mathbf{r}_1 + k\mathbf{r}_2, \qquad k \in Re\,.$$

Now we introduce a three-dimensional Cartesian co-ordinate system by selecting three mutually perpendicular lines through point O and labelling them as the x-, y-, and z-axes in the usual manner (Figure 4.12).

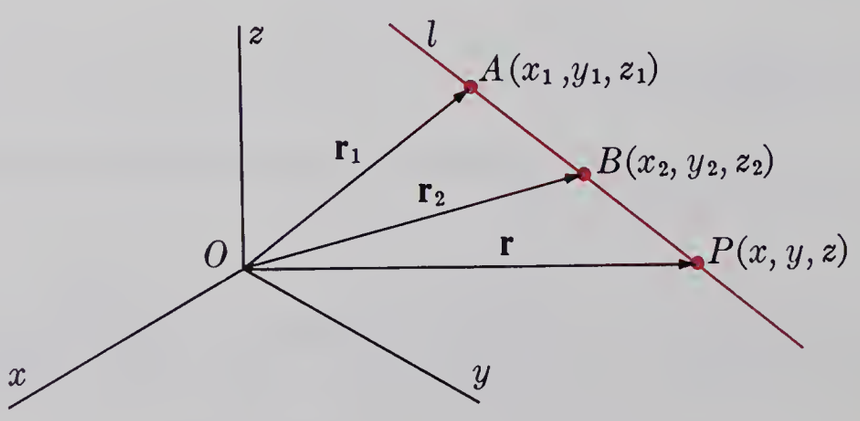


Figure 4.12

Then point A will have co-ordinates (x_1, y_1, z_1) relative to this co-ordinate system and position vector $\mathbf{r}_1$ will be the algebraic vector (x_1, y_1, z_1); thus we will have

$$\mathbf{r}_1 = (x_1, y_1, z_1)\,, \qquad \mathbf{r}_2 = (x_2, y_2, z_2)\,, \qquad \mathbf{r} = (x, y, z)\,.$$

Our vector equation for line l can then be written

$$(x, y, z) = (1 - k)\,(x_1, y_1, z_1) + k(x_2, y_2, z_2)\,.$$

Example 1. Find the vector equation of the line l determined by the points

$$A(-5, 3, 1) \quad \text{and} \quad B(2, 7, -4)\,.$$

Solution: The vector equation of l is

$$(x, y, z) = (1 - k)\,(-5, 3, 1) + k(2, 7, -4)\,.$$

Now the vector equation

$$(x, y, z) = (1 - k)\,(x_1, y_1, z_1) + k(x_2, y_2, z_2)$$

can be written

$$\begin{aligned}(x, y, z) &= [(1 - k)x_1,\ (1 - k)y_1,\ (1 - k)z_1] + (kx_2,\ ky_2, kz_2)\\ &= [(1 - k)x_1 + kx_2,\ (1 - k)y_1 + ky_2,\ (1 - k)z_1 + kz_2]\,.\end{aligned}$$

Identifying components in these two vectors yields

$$\left.\begin{aligned} x &= (1 - k)x_1 + kx_2\\ y &= (1 - k)y_1 + ky_2\\ z &= (1 - k)z_1 + kz_2 \end{aligned}\right\} \quad k \in Re\,.$$

These equations are parametric equations of the line l in 3-space determined by the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. A line in 3-space is represented by a set of three parametric equations.

Again we note that any point on the line determined by points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ has co-ordinates $(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2,\ \alpha z_1 + \beta z_2)$, where $\alpha + \beta = 1$.

Example 2. Find parametric equations of the line l determined by the points $A(-5, 3, 1)$ and $B(2, 7, -4)$ (see Example 1).

Solution: Parametric equations of l are

$$\begin{aligned} x &= (1 - k)\,(-5) + k(2) = -5 + 7k\,,\\ y &= (1 - k)\,(3) + k(7) = 3 + 4k\,,\\ z &= (1 - k)\,(1) + k(-4) = 1 - 5k\,. \end{aligned}$$

As in the corresponding situation in 2-space, any restriction on k in the vector equation or parametric equations of l will yield the equation or equations of a part of l (point, subset of points, line segment, ray, etc.).

Recall also that any particular value of the parameter k determines a point P on l such that $\overrightarrow{AP} = k\overrightarrow{AB}$.

Example 3. Find the point that divides the line segment AB in the ratio 2 : 1 for the points $A(2, 2, -5)$ and $B(-1, 4, 3)$.

Solution: If point P_1 divides line segment AB in the ratio 2:1, then $\overrightarrow{AP_1} = \frac{2}{3}\overrightarrow{AB}$. Now any point P on the line determined by A and B has co-ordinates

$$\begin{aligned} x &= (1 - k)(2) + k(-1) = 2 - 3k, \\ y &= (1 - k)(2) + k(4) = 2 + 2k, \\ z &= (1 - k)(-5) + k(3) = -5 + 8k, \end{aligned}$$

and P_1 is the point corresponding to the value $\frac{2}{3}$ of k. Thus P_1 has co-ordinates

$$[2 - 3(\tfrac{2}{3}),\ 2 + 2(\tfrac{2}{3}),\ -5 + 8(\tfrac{2}{3})] \text{ or } (0, \tfrac{10}{3}, \tfrac{1}{3}).$$

The parametric equations of the line l determined by the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ may be rewritten as

$$x - x_1 = k(x_2 - x_1),\ y - y_1 = k(y_2 - y_1),\ z - z_1 = k(z_2 - z_1),$$

and, by the elimination of k, combined into the equations

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, \quad \text{provided } x_1 \neq x_2,\ y_1 \neq y_2,\ z_1 \neq z_2.$$

These are symmetric equations of the line l in 3-space. (There are essentially two distinct equations involved in this expression; any two of the equations

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, \quad \frac{x - x_1}{x_2 - x_1} = \frac{z - z_1}{z_2 - z_1}, \quad \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

will determine the third.)

Example 4. Find symmetric equations of the line l determined by the points $A(3, -5, 7)$ and $B(-1, -1, 5)$.

Solution: Symmetric equations of l are

$$\frac{x - 3}{-1 - 3} = \frac{y - (-5)}{-1 - (-5)} = \frac{z - 7}{5 - 7};$$

that is,

$$\frac{x - 3}{-4} = \frac{y + 5}{4} = \frac{z - 7}{-2}.$$

(Note that $\frac{x+1}{4} = \frac{y+1}{-4} = \frac{z-5}{2}$ are also symmetric equations of l.)

Parametric equations of l can be found by setting the common ratios equal to k.

$$\frac{x-3}{-4} = \frac{y+5}{4} = \frac{z-7}{-2} = k\,.$$

Hence

$$\begin{aligned} x &= 3 - 4k\,, \\ y &= -5 + 4k\,, \\ z &= 7 - 2k\,. \end{aligned}$$

(Note that $x = -1 + 4k$, $y = -1 - 4k$, $z = 5 + 2k$ are also parametric equations for l.)

Example 5. Find symmetric equations of the line l that passes through the points $A(-3, 1, 7)$ and $B(-4, 1, 5)$.

Solution: It is impossible to write symmetric equations for line l since $y_2 - y_1 = 0$. However, it is possible to obtain parametric equations in the usual way.

EXERCISE 4.6

1. Find vector, parametric, and symmetric (where possible) equations of the line l determined by the following pairs of points.
   (a) $A(1, 1, 1)$, $B(5, 5, 5)$ (b) $A(1, 0, 4)$, $B(5, 3, 9)$
   (c) $A(4, 2, 1)$, $B(1, 2, 4)$ (d) $A(-1, 3, 5)$, $B(7, 7, -3)$
   (e) $A(-4, 1, 7)$, $B(-1, 5, 2)$ (f) $A(4, -3, -3)$, $B(6, 7, -3)$
   (g) $A(5, -1, 3)$, $B(-2, -1, 3)$ (h) $A(-6, 2, 1)$, $B(-1, -1, -7)$
   (i) $A(-3, 1, 6)$, $B(-3, 6, 6)$ (j) $A(0, 0, 1)$, $B(0, 1, 0)$
2. Find the midpoints of the line segments AB determined by the points in question (1).
3. Find the co-ordinates of the point P dividing line segment AB, determined by $A(3, -1, -2)$ and $B(-1, 5, 6)$, in the ratio,
   (a) 1:3, (b) 2:5, (c) -3:2, (d) 5:-1.
4. Locate the following points relative to the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.
   (a) $(\frac{1}{2}x_1 + \frac{1}{2}x_2,\ \frac{1}{2}y_1 + \frac{1}{2}y_2,\ \frac{1}{2}z_1 + \frac{1}{2}z_2)$
   (b) $(\frac{1}{5}x_1 + \frac{4}{5}x_2,\ \frac{1}{5}y_1 + \frac{4}{5}y_2,\ \frac{1}{5}z_1 + \frac{4}{5}z_2)$
   (c) $(-x_1 + 2x_2,\ -y_1 + 2y_2,\ -z_1 + 2z_2)$
   (d) $(-3x_1 + 4x_2,\ -3y_1 + 4y_2,\ -3z_1 + 4z_2)$
   (e) $(2x_1 - x_2,\ 2y_1 - y_2,\ 2z_1 - z_2)$
   (f) $(5x_1 - 4x_2,\ 5y_1 - 4y_2,\ 5z_1 - 4z_2)$

5. Determine the figure with parametric equations

$$x = (1 - k)(-2) + k(4)$$
$$y = (1 - k)(3) + k(-3)$$
$$z = (1 - k)(1) + k(2)$$

for the following sets of values for k.

(a) Re (b) 0 (c) 1 (d) $\frac{1}{2}$ (e) 3
(f) $\{0, 1, \frac{1}{2}\}$ (g) $\{-1, 0, 1\}$
(h) $\{0 \leq k \leq 1,\ k \in Re\}$ (i) $\{k \geq 0,\ k \in Re\}$
(j) $\{k \leq 0,\ k \in Re\}$ (k) $\{k \geq 1,\ k \in Re\} \cup \{k \leq 0,\ k \in Re\}$

6. Describe the location of the lines l with the following parametric equations

(a) $x = 3$, $y = 4 - t$, $z = -1$
(b) $x = 3 + 2t$, $y = 4$, $z = -2$
(c) $x = 3 - 2t$, $y = 4 + 3t$, $z = 2$

7. Find parametric equations of the lines through the point $A(x_1, y_1, z_1)$ and
   (a) parallel to the x-axis,
   (b) parallel to the y-axis,
   (c) parallel to the z-axis.

8. Let the vector equation of line l be $\mathbf{r} = \mathbf{r}_1 + k\mathbf{s}$, where $\mathbf{r}_1$ is the position vector of point P_1 on l, $\mathbf{r}$ is that of an arbitrary point P on l, and $\mathbf{s}$ is a vector parallel to l. If $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{s} = (s_1, s_2, s_3)$, find the corresponding parametric equations of l.

9. Using question (8), find the parametric equations of line l through $P_1(-2, 5, 3)$ if $\mathbf{s}$ is the vector
   (a) $(-3, 1, -4)$, (b) $(4, 0, -7)$, (c) $(0, -2, 1)$.

10. Sketch the parallelogram with the origin and the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ as three of its vertices. Find the equations of the line through A parallel to line OB and those of the line through B parallel to OA. Verify that their point of intersection is $C(x_1 + x_2, y_1 + y_2, z_1 + z_2)$.

11. Sketch the parallelopiped (a parallelopiped is a three-dimensional figure with six faces each of which is a parallelogram) with the origin and the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ as four adjacent vertices. Find the co-ordinates of the other four vertices, A' opposite to A, B' opposite to B, C' opposite to C, and O' opposite to O.

12. Find the co-ordinates of the midpoints of the diagonals AA', BB' CC' and OO' of the parallelopiped in question (11). What conclusion can you make?

4.7. Direction Numbers, Cosines, and Angles of Lines in Three-Space

The work of Sections 4.4 and 4.5 on direction numbers, cosines, and angles of lines in 2-space extends easily to lines in 3-space.

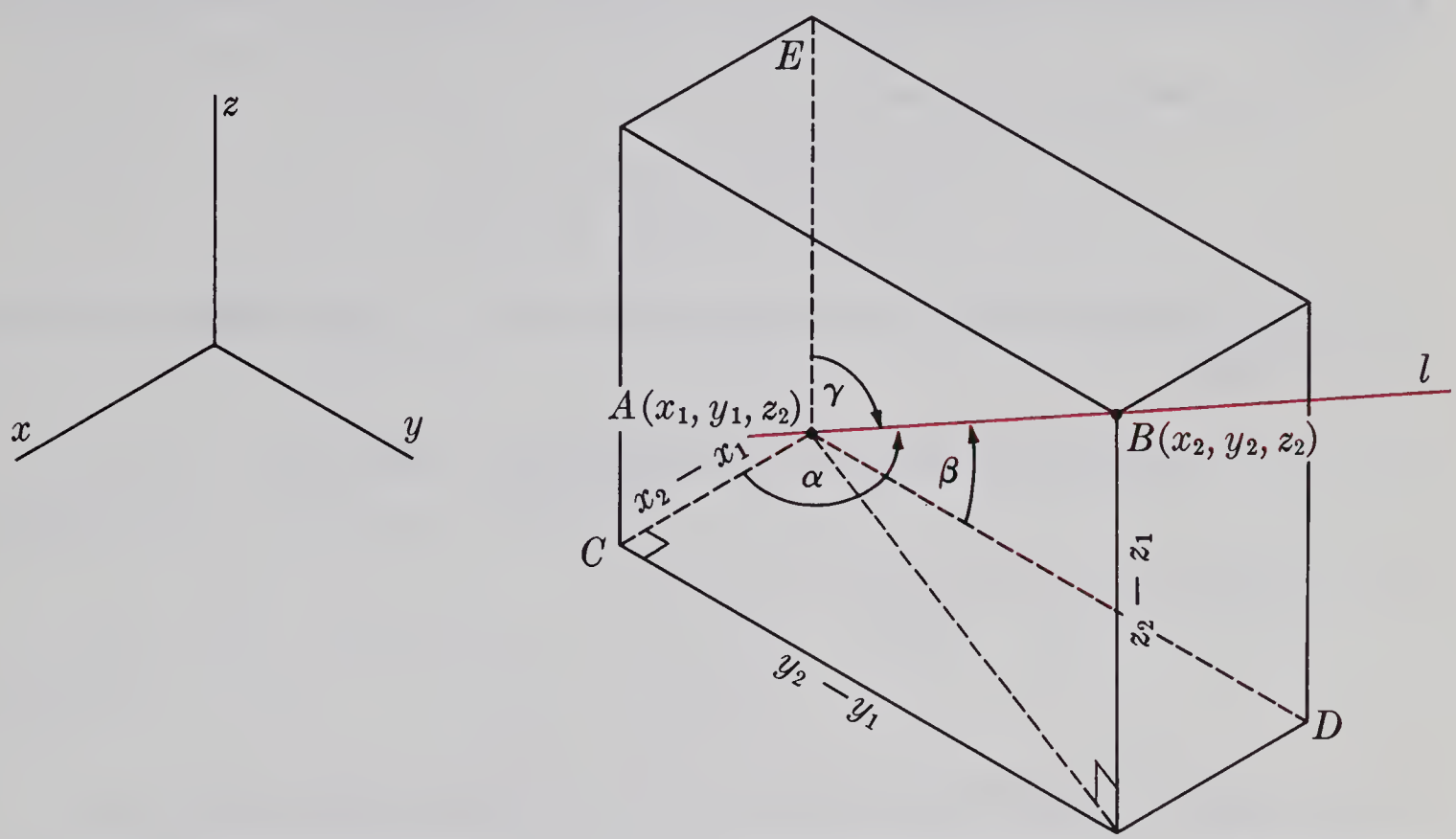


Figure 4.13

The ordered set of numbers $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is a set of direction numbers for the line l through $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ (see Figure 4.13); they determine the direction of line l in the sense that, if we start at any point P in 3-space and travel $x_2 - x_1$ units parallel to the x-axis, $y_2 - y_1$ units parallel to the y-axis, and $z_2 - z_1$ units parallel to the z-axis, we reach a point Q such that PQ is parallel to AB. As in the 2-dimensional case, the ordered set

$$[k(x_2 - x_1)\ ,\ k(y_2 - y_1)\ ,\ k(z_2 - z_1)]$$

is a set of direction numbers for l for any nonzero real number k.

Example 1. Find the direction numbers of the line l through points $A(3, -2, -5)$ and $B(4, 2, -1)$.

Solution: The ordered set

$$(4 - 3, 2 - (-2), -1 - (-5)) \quad \text{or} \quad (1, 4, 4)$$

is a set of direction numbers for l, as is $(k, 4k, 4k)$, for any nonzero real number k.

Note now that an ordered triple (a, b, c) of real numbers can be interpreted as

(1) the co-ordinates of a point A in 3-space,

(2) vector $\overrightarrow{OA}$,

(3) the direction numbers of a family of parallel lines in 3-space (having the direction of vector $\overrightarrow{OA}$).

Again, the specific interpretation in any situation should be determined by the context.

Noting that the denominators in the symmetric equations

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

for the line l form a set of direction numbers for the line l, and that these equations may be written

$$\frac{x - x_1}{k(x_2 - x_1)} = \frac{y - y_1}{k(y_2 - y_1)} = \frac{z - z_1}{k(z_2 - z_1)}, \qquad \text{for } k \neq 0,$$

we see that, in symmetric equations of the form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

(a, b, c) is a set of direction numbers of the line l through $A(x_1, y_1, z_1)$.

Example 2. Find symmetric equations of the line l through the point $A(3, 3, -4)$ with direction numbers $(3, -2, 5)$.

Solution: Symmetric equations of l are

$$\frac{x - 3}{3} = \frac{y - 3}{-2} = \frac{z + 4}{5}.$$

The equations of line l through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ may also be written

$$\frac{x - x_1}{\frac{x_2 - x_1}{d}} = \frac{y - y_1}{\frac{y_2 - y_1}{d}} = \frac{z - z_1}{\frac{z_2 - z_1}{d}},$$

where d is the length of line segment AB. By referring to Figure 4.13, we see that

$$\frac{x_2 - x_1}{d} = \cos\alpha \qquad (\text{from } \triangle ABC, \text{ rt.-angled at } C),$$

$$\frac{y_2 - y_1}{d} = \cos\beta \qquad (\text{from } \triangle ABD, \text{ rt.-angled at } D),$$

$$\frac{z_2 - z_1}{d} = \cos\gamma \qquad (\text{from } \triangle ABE, \text{ rt.-angled at } E),$$

where α, β, and γ are the angles line l (determined by the directed line segment AB) makes with the positive direction of the x-, y-, and z-axes, respectively. Note especially that in general, α, β, and γ lie in three different planes. Angles α, β, γ are called the direction angles of l, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, the direction cosines of l.

Example 3. Find the direction cosines and direction angles, to the nearest degree, of line l through points $P_1(5, 2, 3)$ and $P_2(8, 4, 5)$.

Solution:

$$P_1P_2 = \sqrt{(8-5)^2 + (4-2)^2 + (5-3)^2} = \sqrt{17}\,.$$

$$x_2 - x_1 = 8 - 5 = 3\,, \qquad y_2 - y_1 = 4 - 2 = 2\,, \qquad z_2 - z_1 = 5 - 3 = 2\,,$$

$$\cos\alpha = \frac{3}{\sqrt{17}}\,, \qquad \cos\beta = \frac{2}{\sqrt{17}}\,, \qquad \cos\gamma = \frac{2}{\sqrt{17}}\,,$$

$$\alpha = 44^\circ\,. \qquad \beta = 61^\circ\,. \qquad \gamma = 61^\circ\,.$$

If (a, b, c) is a set of direction numbers for line l, then (see question (4), Exercise 4.7) its direction cosines are

$$\cos\alpha = \frac{a}{d}\,, \qquad \cos\beta = \frac{b}{d}\,, \qquad \cos\gamma = \frac{c}{d}\,, \qquad \text{with } d = \sqrt{a^2 + b^2 + c^2}\,,$$

so that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,.$$

Now there are eight possible patterns of signs for the direction cosines of line l.

$(+, +, +)$	$(-, +, +)$
$(+, +, -)$	$(-, +, -)$
$(+, -, +)$	$(-, -, +)$
$(+, -, -)$	$(-, -, -)$

However, if l has direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, then $-\cos\alpha$, $-\cos\beta$, and $-\cos\gamma$ are the direction cosines of the line in the direction opposite to that of l. As we pointed out in our discussion for 2-space, we do not distinguish between these lines.

We now establish a convention for the choice of sign.

The xy-plane divides 3-space into two halves: an upper half, where z is positive, and a lower half, where z is negative. A line l , not parallel to the xy-plane, has an upper portion, that part of l lying in the upper half of 3-space, and a lower portion, that part lying in the lower half. We then define γ as the angle between the upper portion of l and the positive direction of the z-axis, α as the angle between the upper portion of l and the positive direction of the x-axis, and β as the angle between the upper portion of l and the positive direction of the y-axis. Then $\cos\gamma$ is always positive. We choose γ as the first quadrant angle and α and β as first or second quadrant angles. We make no attempt to assign directions to these angles; such an attempt would be not only difficult in 3-space but fruitless as well, since $\cos\theta = \cos(-\theta)$.

If l is parallel to the xy-plane, we define γ as 90°, α in the same way as for 2-space and β as the *unsigned* angle between the upper portion of l and the positive direction of the y-axis. In this case, we define the upper portion of l as that part of l directly above or below the upper portion of l', where l' is the line parallel to l and lying in the xy-plane.

Example 4. Find the direction angles of line l if its direction cosines, without regard to our convention on signs, are given as $-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$.

Solution: We must take the negative of these direction numbers to get one of our accepted patterns for direction cosines; we take

$$\begin{aligned} \cos\alpha &= \tfrac{1}{3} & \cos\beta &= -\tfrac{2}{3} & \cos\gamma &= \tfrac{2}{3} \\ &\simeq 0.333. & &\simeq -0.667. & &\simeq 0.667. \\ \alpha &= 71°. & \beta &= 132°. & \gamma &= 48°. \end{aligned}$$

EXERCISE 4.7

1. Find the direction numbers, cosines, and angles (to the nearest degree) of the following lines.
   (a) Line l through $A(2, 5, 2)$ and $B(3, 6, 3)$
   (b) Line l through $A(0, 0, 4)$ and $B(5, 1, 5)$
   (c) Line l through $A(3, -2, 5)$ and $B(4, -2, 6)$
   (d) Line l through $A(0, 0, 1)$ and $B(1, 0, 0)$
   (e) Line l with equations $\dfrac{x-3}{4} = \dfrac{y+2}{-1} = \dfrac{z-3}{5}$
   (f) Line l with equations $\dfrac{x+2}{-5} = \dfrac{y}{-4} = \dfrac{z+1}{3}$
   (g) Line l with equations $\dfrac{2x-3}{4} = \dfrac{3y+2}{-1} = \dfrac{z-3}{2}$
   (h) Line l with equations $x = 4 - 3t,\ y = -2 + t,\ z = 4t$
   (i) Line l with equations $x = 3 - 2t,\ y = -6,\ z = 2 + 3t$
   (j) Line l with equations $x = 5 + 3t,\ y = 1 - 2t,\ z = 3$
2. What can be said about the line l if its direction numbers are
   (a) $(0, 0, k)$? (b) $(0, k, 0)$? (c) $(k, 0, 0)$?
3. Find the equations of the following lines.
   (a) l through $A(1, 1, 5)$ with direction numbers $(4, 3, -1)$
   (b) l through $A(-3, 1, 3)$ with direction numbers $(-1, 2, -4)$
   (c) l through $A(4, 2, 4)$ with direction numbers $(3, 0, -2)$
   (d) l through $A(-1, 5, 2)$ with direction numbers $(0, 3, 0)$
   (e) l through $A(4, 1, 7)$ with direction cosines $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$
   (f) l through $A(2, -1, 8)$ with direction cosines $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)$

(g) l through $A(4, 1, 0)$ with direction angles 45°, 45°, 90°

(h) l through $A(-2, 1, 2)$ with direction angles 60°, 120°, 45°

4. Show that if (a, b, c) is a set of direction numbers for line l, then its direction cosines, without regard to our convention on signs, are given by

$$\cos\alpha = \frac{a}{d}, \qquad \cos\beta = \frac{b}{d}, \qquad \cos\gamma = \frac{c}{d}, \qquad \text{where } d = \sqrt{a^2 + b^2 + c^2}.$$

Chapter Summary

Vector equation of a line · Positional vector equation of a line · Parametric equations of a line in 2-space (from the vector equation) · The linear equation of a line in 2-space · Direction angles, cosines, and numbers of lines in 2-space · Equations of a line in 3-space · Direction angles, cosines, and numbers of lines in 3-space

REVIEW EXERCISE 4

1. If $\overrightarrow{AP} = k\overrightarrow{AB}$, determine P relative to points A and B for the following real values of k (include a diagram).

(a) $\frac{3}{4}$ (b) $-\frac{4}{3}$ (c) $\frac{1}{2} \leq k \leq 1$ (d) $k \leq -1$

2. In question (1), determine k if

(a) P divides line segment AB in the ratio 1:4,

(b) P is three times as far from B as from A.

3. If $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are position vectors of points P, A, and B with respect to a fixed point O, interpret the equation

$$\mathbf{r} = (1 - k)\mathbf{r}_1 + k\mathbf{r}_2$$

for the following sets of real values for k.

(a) $k = \{0, 1\}$ (b) $k \geq 1$ (c) $-1 \leq k \leq 1$

4. Give vector and parametric equations for lines determined by pairs of points with the following co-ordinates (include a diagram).

(a) $(4, 1)$ and $(3, 4)$ (b) $(-2, 1)$ and $(3, -2)$

(c) $(5, 0)$ and $(0, 2)$ (d) $(-3, -4)$ and $(-2, 0)$

5. Describe the figure with parametric equations

$$x = (1 - k)(-3) + k(4)$$
$$y = (1 - k)(5)$$

for the following sets of real values of k.

(a) $k = \{0, 1\}$ (b) $k \geq 0$ (c) $k \in Re$

6. Write vector and parametric equations for the line l passing through point A and collinear with the vector $\mathbf{s}$ for the following.
   (a) $A(2, -1)$, $\mathbf{s} = (3, -1)$ (b) $A(5, 0)$, $\mathbf{s} = (-2, -3)$

7. The normal axis of line l lies on point M; l lies on point Q. Find the linear equation of l in the following cases.
   (a) $M(2, 3)$, $Q(4, -1)$ (b) $M(3, 0)$, $Q(-2, -3)$
   (c) $M(5, -2)$, $Q(0, 0)$ (d) $M(4, 4)$, $Q(3, 3)$

8. Find the equation of line l if l and its normal axis intersect in the following points.
   (a) $N(5, 2)$ (b) $N(-3, 1)$ (c) $N(-4, -3)$

9. Find the intersection of line l and its normal axis l', and sketch the lines l with the following equations.
   (a) $x + 4y = 9$ (b) $2x = 7$
   (c) $3x - 2y = -4$ (d) $-x + 3y = -6$

10. Find the distance from the origin to line l in question (9).

11. Find the distance from point $R(-1, 5)$ to line l in question (9).

12. Find (i) direction numbers, (ii) direction cosines, (iii) direction angles, (iv) symmetric equations of the following lines.
   (a) Line l through $P_1(-3, 4)$ and $P_2(4, -3)$
   (b) Line l with equation $6x - 4y = 24$
   (c) Line l with parametric equations $x = 5 - 2k$, $y = -3 + k$
   (d) Line l through $P_1(5, -2)$ if its normal axis lies on $M(3, 4)$

13. Find vector, parametric, and symmetric equations (where possible) of the lines l determined by the following pairs of points.
   (a) $A(3, -2, 3)$, $B(4, 0, -2)$ (b) $A(5, 2, -1)$, $B(5, 2, 6)$
   (c) $A(-1, 3, -2)$, $B(5, -3, -3)$ (d) $A(2, 1, 7)$, $B(-6, 1, -4)$

14. Find the midpoints of the line segments AB determined by the points in question (13).

15. Find the direction numbers, cosines, and angles, to the nearest degree, of the lines in question (13).

16. Find the direction numbers, cosines, and angles, to the nearest degree, of the following lines.
   (a) Line l with equations $\dfrac{2x - 5}{3} = \dfrac{y + 2}{-7} = \dfrac{z - 6}{3}$
   (b) Line l with parametric equations $x = 3 - 5t$, $y = 4 - t$, $z = -3$

17. Find the equations of the following lines.
   (a) l through $A(4, 1, -2)$ with direction numbers $(5, 1, -4)$
   (b) l through $A(-2, 3, -1)$ with direction cosines $(\frac{3}{5}, 0, \frac{4}{5})$
   (c) l through $A(5, -1, 2)$ with direction angles $60°$, $45°$, $120°$

Chapter 5

PLANES

5.1. The Vector Equation of the Plane

Let a plane I be determined by three noncollinear points A, B, and C, and let P be an arbitrary point on the plane (Figure 5.1). Let PD be parallel to AC and PE be parallel to AB. Then, $\overrightarrow{AD} = k\,\overrightarrow{AB}$ and $\overrightarrow{AE} = m\,\overrightarrow{AC}$ and

$$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{AE},$$

so that

$$\overrightarrow{AP} = k\,\overrightarrow{AB} + m\,\overrightarrow{AC}, \qquad k,m \in Re$$

is a vector representation or equation of the plane I determined by the noncollinear points A, B, and C.

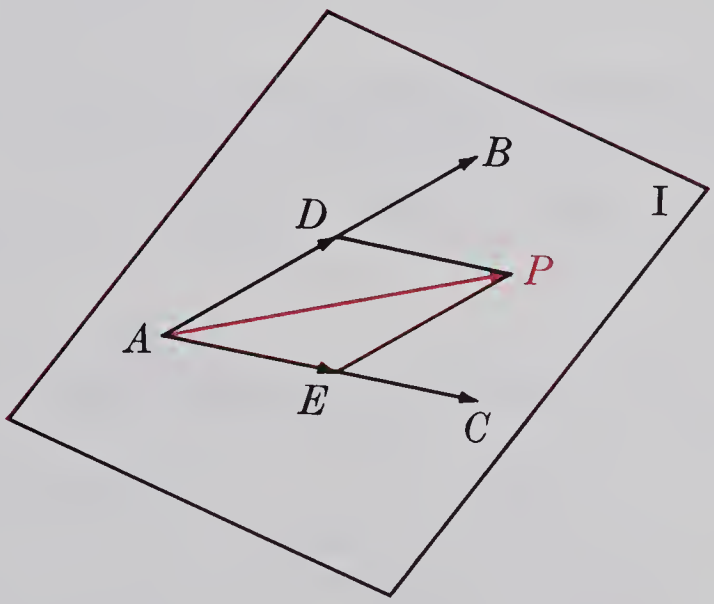


Figure 5.1

Now choose a point O in 3-space. With regard to this point, any point P will have a position vector $\overrightarrow{OP}$ represented by the line segment from O to P; similarly, points A, B, and C will have position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$, respectively (Figure 5.2).

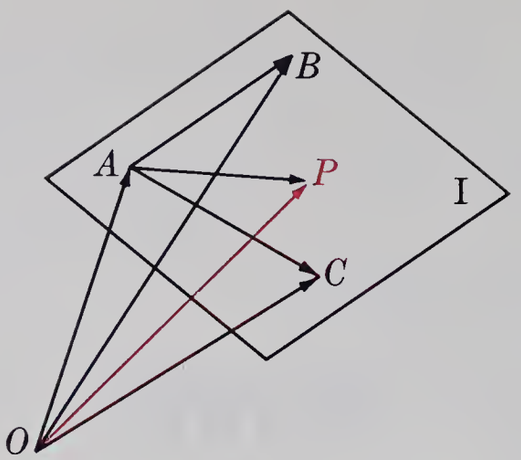


Figure 5.2

Then,

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \overrightarrow{OA} + k\,\overrightarrow{AB} + m\,\overrightarrow{AC} \qquad (k,m \in Re) \\ &= \overrightarrow{OA} + k(\overrightarrow{OB} - \overrightarrow{OA}) + m(\overrightarrow{OC} - \overrightarrow{OA}),\end{aligned}$$

so that

$$\overrightarrow{OP} = (1 - k - m)\overrightarrow{OA} + k\,\overrightarrow{OB} + m\,\overrightarrow{OC}, \qquad k,m \in Re\,,$$

is the positional vector equation of the plane I in terms of the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ of the points A, B, and C determining I. If we denote vector $\overrightarrow{OP}$ by $\mathbf{r}$, $\overrightarrow{OA}$ by $\mathbf{r}_1$, $\overrightarrow{OB}$ by $\mathbf{r}_2$, and $\overrightarrow{OC}$ by $\mathbf{r}_3$, this equation becomes

$$\mathbf{r} = (1 - k - m)\mathbf{r}_1 + k\,\mathbf{r}_2 + m\,\mathbf{r}_3\,.$$

Example. Interpret the equation $\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{1}{3}\mathbf{r}_2 + \frac{1}{3}\mathbf{r}_3$.

Solution: $\mathbf{r}$ is the position vector of a point P on the plane ABC determined by points A, B, and C with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; P is such that

$$\overrightarrow{AP} = k\,\overrightarrow{AB} + m\,\overrightarrow{AC} \qquad \text{with } k = m = \tfrac{1}{3} \text{ (Figure 5.3).}$$

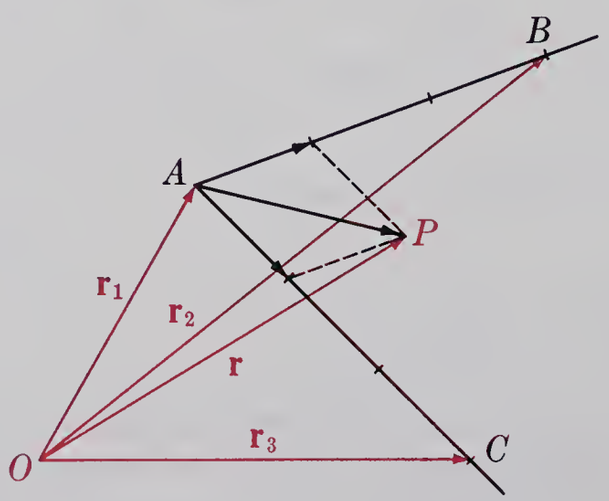


Figure 5.3

If, in the development of the positional vector equation of the plane determined by the noncollinear points A, B, and C, we let $\mathbf{s} = \overrightarrow{AB}$ and $\mathbf{t} = \overrightarrow{AC}$, we obtain

$$\mathbf{r} = \mathbf{r}_1 + k\mathbf{s} + m\mathbf{t}\,.$$

This equation is another useful form of the vector equation of a plane.

EXERCISE 5.1

1. If noncollinear points A, B, and C have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, locate points P with the position vectors given below. (Use a diagram.)
   (a) $\mathbf{r} = \frac{1}{2}\mathbf{r}_1 + \frac{1}{4}\mathbf{r}_2 + \frac{1}{4}\mathbf{r}_3$ (b) $\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{2}{3}\mathbf{r}_2$
   (c) $\mathbf{r} = \frac{4}{5}\mathbf{r}_1 + \frac{1}{5}\mathbf{r}_3$ (d) $\mathbf{r} = 3\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3$
   (e) $\mathbf{r} = 4\mathbf{r}_1 - 2\mathbf{r}_2 - \mathbf{r}_3$ (f) $\mathbf{r} = \mathbf{r}_1 + 2\mathbf{r}_2 - 2\mathbf{r}_3$
   (g) $\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{1}{4}\mathbf{r}_2 + \frac{5}{12}\mathbf{r}_3$ (h) $\mathbf{r} = \frac{3}{2}\mathbf{r}_1 - \mathbf{r}_2 + \frac{1}{2}\mathbf{r}_3$
2. Find k and m in the following equations so that $\mathbf{r}$ is the position vector of a point coplanar with A, B, and C.
   (a) $\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{1}{6}\mathbf{r}_2 + m\mathbf{r}_3$ (b) $\mathbf{r} = 2\mathbf{r}_1 + k\mathbf{r}_2 + 3\mathbf{r}_3$
   (c) $\mathbf{r} = -2\mathbf{r}_1 + k\mathbf{r}_2 + m\mathbf{r}_3$ (d) $\mathbf{r} = 4\mathbf{r}_1 + k\mathbf{r}_2 - m\mathbf{r}_3$
3. Describe the region determined by equation

$$\mathbf{r} = (1 - k - m)\mathbf{r}_1 + k\mathbf{r}_2 + m\mathbf{r}_3$$

if (a) $0 \le k \le 1$, $0 \le m \le 1$, $k,m \in Re$,
(b) $0 \le k \le 1$, $k,m \in Re$,
(c) $0 \le m \le 1$, $k,m \in Re$,
(d) $0 \le k \le 1$, $-1 \le m \le 0$, $k,m \in Re$,
(e) $-1 \le k \le 0$, $-1 \le m \le 0$, $k,m \in Re$,
(f) $k \ge 0$, $m \le 0$, $k, m \in Re$.

5.2. The Parametric Equations of a Plane

We now have the vector equation

$$\mathbf{r} = (1 - k - m)\mathbf{r}_1 + k\mathbf{r}_2 + m\mathbf{r}_3, \quad k,m \in Re\,,$$

for the plane I determined by three noncollinear points A, B, and C with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, with respect to a fixed point O. $\mathbf{r}$ is then the position vector of an arbitrary point P on the plane (depending on the real values assigned to the parameters k and m).

Suppose now that we introduce a three-dimensional Cartesian co-ordinate system in the usual way with origin at O (Figure 5.4).

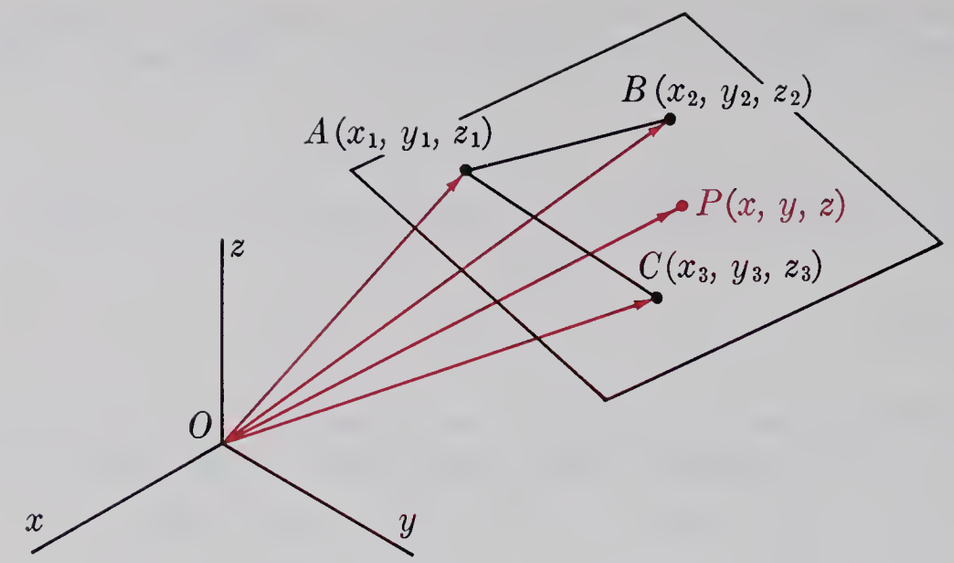


Figure 5.4

Then, point A will have co-ordinates (x_1, y_1, z_1) relative to this co-ordinate system, and its position vector $\mathbf{r}_1$ will be the algebraic vector (x_1, y_1, z_1); thus, for A, $\mathbf{r}_1 = (x_1, y_1, z_1)$, for B, $\mathbf{r}_2 = (x_2, y_2, z_2)$, for C, $\mathbf{r}_3 = (x_3, y_3, z_3)$, for P, $\mathbf{r} = (x, y, z)$. The vector equation for the plane I may be written

$$(x, y, z) = (1 - k - m)(x_1, y_1, z_1) + k(x_2, y_2, z_2) + m(x_3, y_3, z_3) .$$

Example 1. Find the vector equation of the plane I determined by the points $A(-5, 3, 1)$, $B(2, 7, -4)$, and $C(5, 0, -2)$.

Solution: The vector equation may be written

$$(x, y, z) = (1 - k - m)(-5, 3, 1) + k(2, 7, -4) + m(5, 0, -2) .$$

This equation may be written in the form

$$\begin{aligned}&(x, y, z)\\ &= [-5(1 - k - m), 3(1 - k - m), (1 - k - m)] + (2k, 7k, -4k) + (5m, 0, -2m)\\ &= (-5 + 7k + 10m, 3 + 4k - 3m, 1 - 5k - 3m) .\end{aligned}$$

In general, the vector equation

$$(x, y, z) = (1 - k - m)(x_1, y_1, z_1) + k(x_2, y_2, z_2) + m(x_3, y_3, z_3)$$

may be written in the form

$$\begin{aligned}(x, y, z) &= [(1 - k - m)x_1, (1 - k - m)y_1, (1 - k - m)z_1] + (kx_2, ky_2, kz_2)\\ &\quad + (mx_3, my_3, mz_3)\\ &= [(1 - k - m)x_1 + kx_2 + mx_3, (1 - k - m)y_1 + ky_2 + my_3,\\ &\quad (1 - k - m)z_1 + kz_2 + mz_3] .\end{aligned}$$

Identifying corresponding components in these two vectors yields

$$\begin{aligned} x &= (1-k-m)\,x_1 + kx_2 + mx_3\,, \\ y &= (1-k-m)\,y_1 + ky_2 + my_3\,, \\ z &= (1-k-m)\,z_1 + kz_2 + mz_3\,, \qquad k, m \in Re\,. \end{aligned}$$

These are the parametric equations of the plane determined by the three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$. Note that the equations involve two parameters, k and m. Also note that every point on the plane has co-ordinates

$$(\alpha x_1 + \beta x_2 + \gamma x_3,\ \alpha y_1 + \beta y_2 + \gamma y_3,\ \alpha z_1 + \beta z_2 + \gamma z_3)\,,$$

where $\alpha + \beta + \gamma = 1$.

Example 2. Find the parametric equations of the plane I determined by the points $A(-5, 3, 1)$, $B(2, 7, -4)$, and $C(5, 0, -2)$ (see Example 1).

Solution: Identifying vector components from the vector equation of Example 1 yields the parametric equations of the plane

$$\begin{aligned} x &= -5 + 7k + 10m\,, \\ y &= 3 + 4k - 3m\,, \\ z &= 1 - 5k - 3m\,. \end{aligned}$$

These can also be obtained by direct substitution in the parametric equations for the plane.

It should be pointed out that any restrictions on k and m in the parametric equations of the plane will give the equations of a *part* of the plane (a point, a line, a region, etc.).

EXERCISE 5.2

1. Find vector and parametric equations of the planes determined by the following sets of points.
   (a) $A(0, 0, 0)$, $B(1, 4, 9)$, $C(2, 2, 5)$
   (b) $A(0, 0, 0)$, $B(-1, -3, -1)$, $C(2, 5, -4)$
   (c) $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$
   (d) $A(0, 0, 3)$, $B(0, -2, 0)$, $C(5, 0, 0)$
   (e) $A(5, 0, 0)$, $B(3, 2, -1)$, $C(4, 4, 0)$
   (f) $A(-1, 3, -1)$, $B(0, 2, 0)$, $C(3, -1, 2)$
   (g) $A(2, 4, 3)$, $B(2, -1, 5)$, $C(2, 0, 6)$

2. Describe the location of the planes with the following parametric equations.

(a) $x = -5$, $y = 4 + k + m$, $z = 1 - 2k - m$

(b) $x = 4 - k + 2m$, $y = 3$, $z = -2 + k - m$

(c) $x = -2 + 3k - 4m$, $y = 3 - 2k - m$, $z = 0$

3. Find the parametric equations of the planes through the point $A(x_1, y_1, z_1)$ and

(a) parallel to the yz-plane,

(b) parallel to the xz-plane,

(c) parallel to the xy-plane.

4. Describe the figure with parametric equations

$$x = (1 - k - m)(4) + k(-3) + m,$$
$$y = (1 - k - m)(-2) + k(5) + m(2),$$
$$z = (1 - k - m)(3) + k(-1) + m(-1),$$

if $k, m \in Re$ and $0 \leq k \leq 1$, $0 \leq m \leq 1$.

5.3. The Linear Equation of the Plane

We may obtain the linear equation of the plane in 3-space by making the following observation. A plane I in 3-space (Figure 5.5) determines a unique line l through the origin and perpendicular to I; we call this line the normal axis of I. The normal axis has the direction of vector $\overrightarrow{OM}$ joining the origin O to point $M(a, b, c)$ on l.

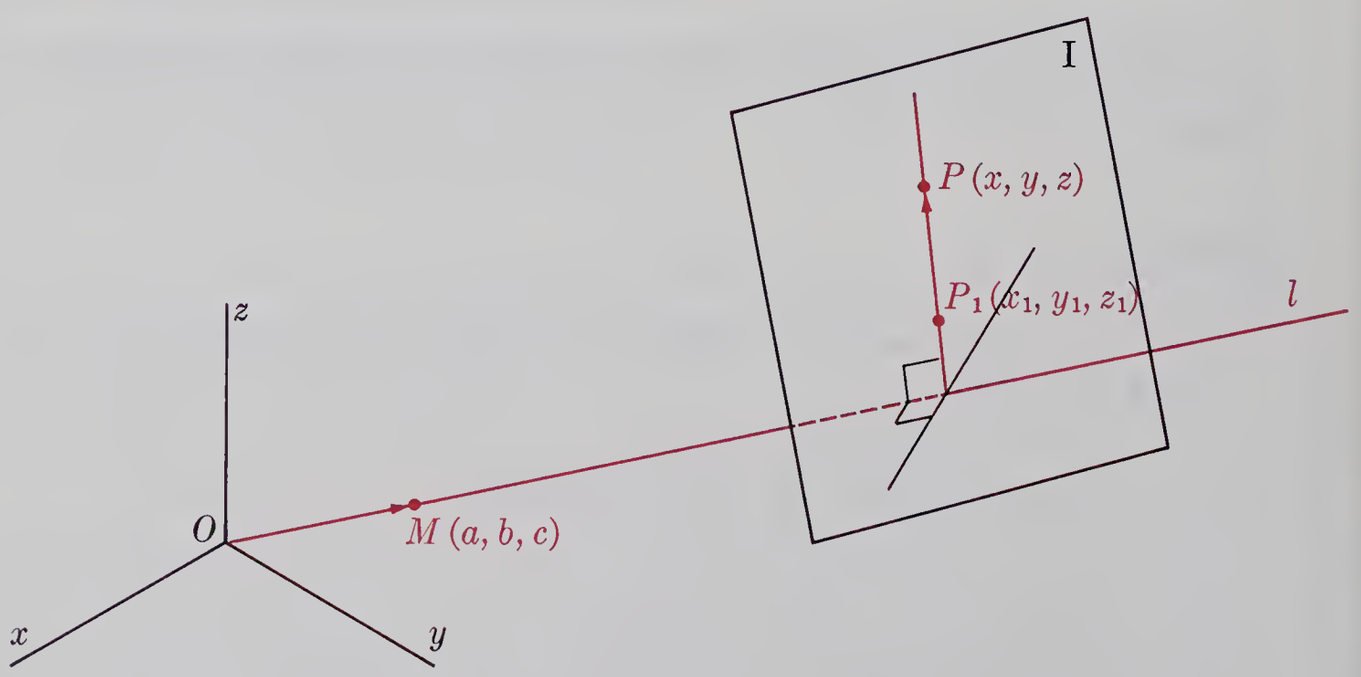


Figure 5.5

A particular plane in 3-space is thus determined when we know a point $P_1(x_1, y_1, z_1)$ on the plane and the direction of the normal axis. (This is known when we know a point $M(a, b, c)$ on the normal axis.) We obtain an equation for the plane by noting that, if $P(x, y, z)$ is any point on the plane, $P \neq P_1$, then vector $\overrightarrow{P_1P}$ is perpendicular to vector $\overrightarrow{OM}$.

Theorem. Let plane I be determined by

(i) point $P_1(x_1, y_1, z_1)$ on I ,

(ii) point $M(a, b, c)$ on l, the normal axis of I .

Then I has the equation

$$ax + by + cz = ax_1 + by_1 + cz_1 .$$

Proof: Let $P(x, y, z)$ be any point on I, $P \neq P_1$. Then

$$\overrightarrow{P_1P} = (x - x_1, y - y_1, z - z_1)$$

is perpendicular to

$$\overrightarrow{OM} = (a, b, c) .$$

But $\overrightarrow{OM} \perp \overrightarrow{P_1P}$ if and only if

$$\overrightarrow{OM} \cdot \overrightarrow{P_1P} = 0 .$$

$$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0 ;$$

that is,

$$ax - ax_1 + by - by_1 + cz - cz_1 = 0 ,$$

or

$$ax + by + cz = ax_1 + by_1 + cz_1 .$$

Note that the equation of a plane is of the form

$$ax + by + cz = d , \qquad a, b, c, d \in Re ,$$

where (a, b, c) is a vector perpendicular to the plane.

Example 1. Find the equation of the plane perpendicular to the vector $\mathbf{v} = (2, -3, -3)$ and containing the point $P_1(1, 1, 2)$.

Solution:

First Method: Let $P(x, y, z)$ be any point on the plane, $P \neq P_1$. Then $\overrightarrow{P_1P} = (x - 1, y - 1, z - 2)$ is perpendicular to $\mathbf{v} = (2, -3, -3)$.

$$(2, -3, -3) \cdot (x - 1, y - 1, z - 2) = 0 ;$$

that is,

$$2x - 2 - 3y + 3 - 3z + 6 = 0$$

or

$$2x - 3y - 3z = -7 .$$

Second Method: Since the vector $\mathbf{v} = (2, -3, -3)$ is perpendicular to the plane, the plane has an equation of the form

$$2x - 3y - 3z = d\,.$$

Since $P_1(1, 1, 2)$ lies on the plane, we have

$$2(1) - 3(1) - 3(2) = d\,.$$

Thus,

$$d = -7\,.$$

The required equation is $2x - 3y - 3z = -7$, as before.

It is sometimes useful to choose as the point P_1 on the plane I the point N of intersection of the plane and its normal axis (Figure 5.6). N will have co-ordinates (ka, kb, kc), since $\overrightarrow{ON} = k\overrightarrow{OM}$ for some real number k. Then, if $P(x, y, z)$ is any different point on I,

$$\overrightarrow{P_1P} = (x - ka,\ y - kb,\ z - kc)$$

and

$$\overrightarrow{OM} \perp \overrightarrow{P_1P}$$

so that

$$(a, b, c) \cdot (x - ka,\ y - kb,\ z - kc) = 0\,,$$

and this equation yields

$$ax + by + cz = k(a^2 + b^2 + c^2)$$

as an alternative form of the equation of the plane.

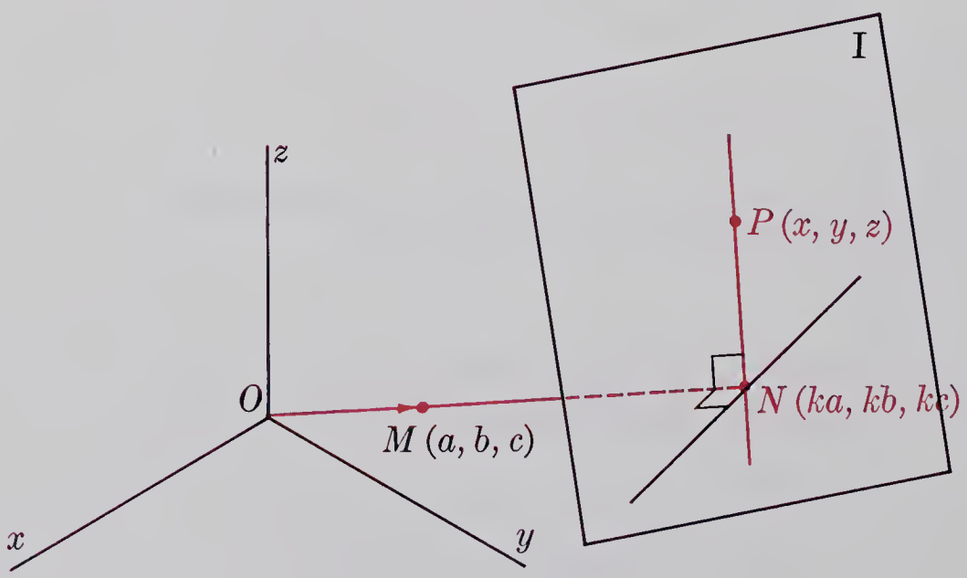


Figure 5.6

When the equation of the plane is written in this form, we can obtain immediately the normal intercept p of the plane, that is, the perpendicular distance from the origin to the plane, or the distance cut off on the normal axis by the plane. This distance is the length of vector $\overrightarrow{ON}$ (equivalently the length of line segment ON), so that

$$p = |\overrightarrow{ON}| = |k|\sqrt{a^2 + b^2 + c^2}\,.$$

Example 2. Find the equation of the plane perpendicular to the vector $(2, -1, 3)$ and with normal intercept 5 (two answers).

Solution: We may write the equation of the plane in the form

$$\begin{aligned} 2x - y + 3z &= k(4 + 1 + 9) \\ &= 14k\,. \end{aligned}$$

This plane has normal intercept $|k|\sqrt{14}$ and so

$$\begin{aligned} |k|\sqrt{14} &= 5\,. \\ k\sqrt{14} &= 5 \quad \text{or} \quad -5\,. \\ 14k &= 5\sqrt{14} \quad \text{or} \quad -5\sqrt{14}\,. \end{aligned}$$

The required equations are $2x - y + 3z = 5\sqrt{14}$ and $2x - y + 3z = -5\sqrt{14}$.

Example 3. The plane I has equation $2x + y - 5z = 16$; find

(a) a vector perpendicular to I,

(b) a point on the normal axis l of I,

(c) the point of intersection of l and I,

(d) the distance from O to I.

Solution:

(a) Vector $(2, 1, -5)$ is perpendicular to I.

(b) Point $M(2, 1, -5)$ lies on the normal axis l.

(c) Writing $2x + y - 5z = 16$ in the form $ax + by + cz = k(a^2 + b^2 + c^2)$, we obtain

$$2x + y - 5z = \tfrac{16}{30}(2^2 + 1^2 + (-5)^2)\,,$$

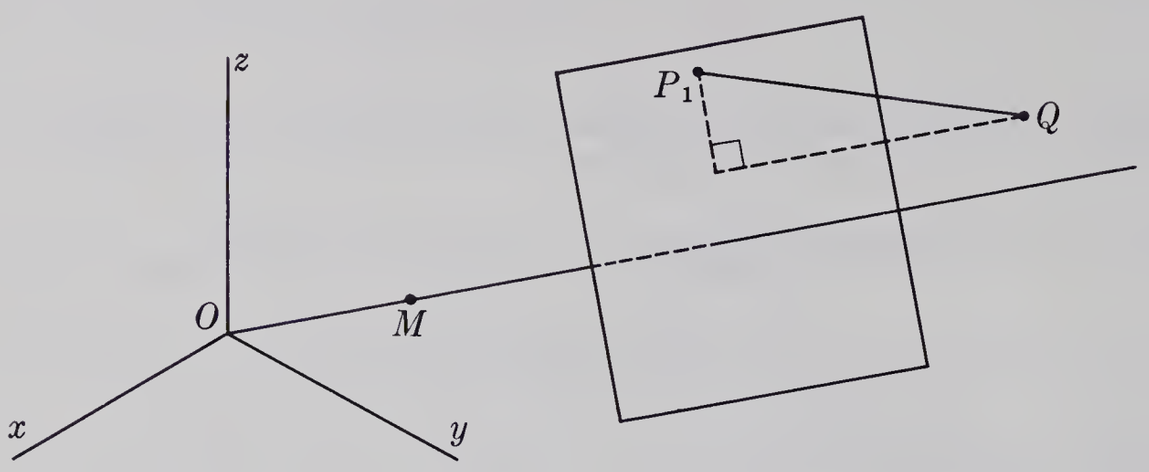
so that $k = \frac{8}{15}$. Then the point of intersection N of I and l is such that $\overrightarrow{ON} = \frac{8}{15}\overrightarrow{OM}$; thus N has co-ordinates $(\frac{16}{15}, \frac{8}{15}, -\frac{8}{3})$.

(d) The distance from O to I is $|k|\sqrt{a^2 + b^2 + c^2}$, that is, in this example, $\frac{8}{15}\sqrt{30}$.

Example 4. Find the distance from the point $Q(4, -1, 7)$ to the plane I, whose equation is

$$2x + y - 5z = 16\,.$$

Solution:



If P_1 is a point on I, the required distance is the projection of the vector $\overrightarrow{QP_1}$ on a vector perpendicular to I; $\overrightarrow{OM} = (2, 1, -5)$ is such a vector.

It is a simple matter to find the co-ordinates of a point P_1 on I; assign y and z arbitrary values and solve for x. If $y = z = 0$, then $x = 8$, so that $(8, 0, 0)$ may be taken as co-ordinates for P_1. Then $\overrightarrow{QP_1} = (4, 1, -7)$ and

$$\begin{aligned}\text{required distance} &= \text{projection of } \overrightarrow{QP_1} \text{ on } \overrightarrow{OM} \\ &= \frac{|(4, 1, -7) \cdot (2, 1, -5)|}{|\,(2, 1, -5)\,|} \\ &= \frac{44}{\sqrt{30}}\,.\end{aligned}$$

EXERCISE 5.3

1. Find the equations of the planes perpendicular to the given vectors $\mathbf{v}$ and passing through the given points P_1.

   (a) $\mathbf{v} = (5, 5, 1)$, $P_1(1, 1, 5)$ (b) $\mathbf{v} = (0, 1, 2)$, $P_1(3, -1, 4)$

   (c) $\mathbf{v} = (5, 2, 0)$, $P_1(-3, 1, 5)$ (d) $\mathbf{v} = (2, 0, 0)$, $P_1(3, 3, 1)$

   (e) $\mathbf{v} = (4, 2, -1)$, $P_1(1, 0, 1)$ (f) $\mathbf{v} = (-3, -2, 2)$, $P_1(5, 0, 2)$

   (g) $\mathbf{v} = (3, 3, -4)$, $P_1(0, 0, 0)$ (h) $\mathbf{v} = (4, 0, 0)$, $P_1(0, 0, 0)$

   (i) $\mathbf{v} = (3, 5, -2)$, $P_1(1, 1, 1)$ (j) $\mathbf{v} = (0, -1, 0)$, $P_1(5, 0, 4)$

2. Find the equations of the planes perpendicular to the following vectors and with the given normal intercepts (two solutions for each).

   (a) $(0, 2, 0)$, 4 (b) $(-3, 0, 0)$, 2 (c) $(5, 0, -3)$, 3

   (d) $(0, 5, -2)$, $\frac{1}{2}$ (e) $(3, -3, 1)$, $\sqrt{3}$ (f) $(2, -3, -5)$, π

3. Find the equations of the planes which meet their normal axes in the following points.

   (a) $(2, 2, 3)$ (b) $(-4, 3, 0)$ (c) $(4, -1, -3)$

4. Find the points of intersection with their normal axes of the planes with the following equations.

   (a) $2x - 3y + z = 0$ (b) $x + 3y + 4z = 5$ (c) $-x + 3y + z = 10$

   (d) $5x - y - z = 16$ (e) $-2x + 3y - z = 24$ (f) $3x - y + 4z = -16$

5. Find the distance from the origin to the planes in question (4).

6. Prove that the distance from the origin to the plane whose equation is $ax + by + cz = d$ is

$$\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

7. Find the distance from the following points Q to the plane whose equation is given.

   (a) $Q(4, 0, 0)$, $3x - 2y + z = 0$ (b) $Q(1, 1, 0)$, $2x - 3y + z = 6$

   (c) $Q(-1, 2, 4)$, $2x + 5y - 3z = 30$ (d) $Q(2, 2, -3)$, $x + 4y - z = 16$

   (e) $Q(0, 4, -3)$, $-x - 2y + 3z = 12$ (f) $Q(-1, 5, -1)$, $2x - 3y - 4z = 24$

8. Prove that the distance from the point $Q(x_1, y_1, z_1)$ to the plane whose equation is $ax + by + cz = d$ is

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

9. Using the formula

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

   for the distance from the point $Q(x_1, y_1, z_1)$ to the plane with equation $ax + by + cz = d$, verify your answers in questions (6) and (7).

5.4. Sketching Planes in Three-Space

Recall that, through any three distinct noncollinear points, there is a unique plane. To graph $ax + by + cz = d$, we find three distinct noncollinear points whose co-ordinates satisfy the given equation and use the triangle formed by the line segments joining these points to indicate the entire plane they determine.

Example 1. Graph the equation $x + 2y + 3z = 6$.

Solution: All we need do is find three noncollinear points on the locus. In this example, we can proceed by finding the points where the locus meets the co-ordinate axes. We proceed as follows.

(i) Let $y = z = 0$ in the given equation; this leaves $x = 6$, and thus $(6, 0, 0)$ is a point on the locus which is also on the x-axis, and hence is the point of intersection with the x-axis.

(ii) Let $x = z = 0$; this leaves $2y = 6$ or $y = 3$, and so $(0, 3, 0)$ is the point of intersection with the y-axis.

(iii) Let $x = y = 0$; this leaves $3z = 6$ or $z = 2$ and so $(0, 0, 2)$ is the point of intersection with the z-axis.

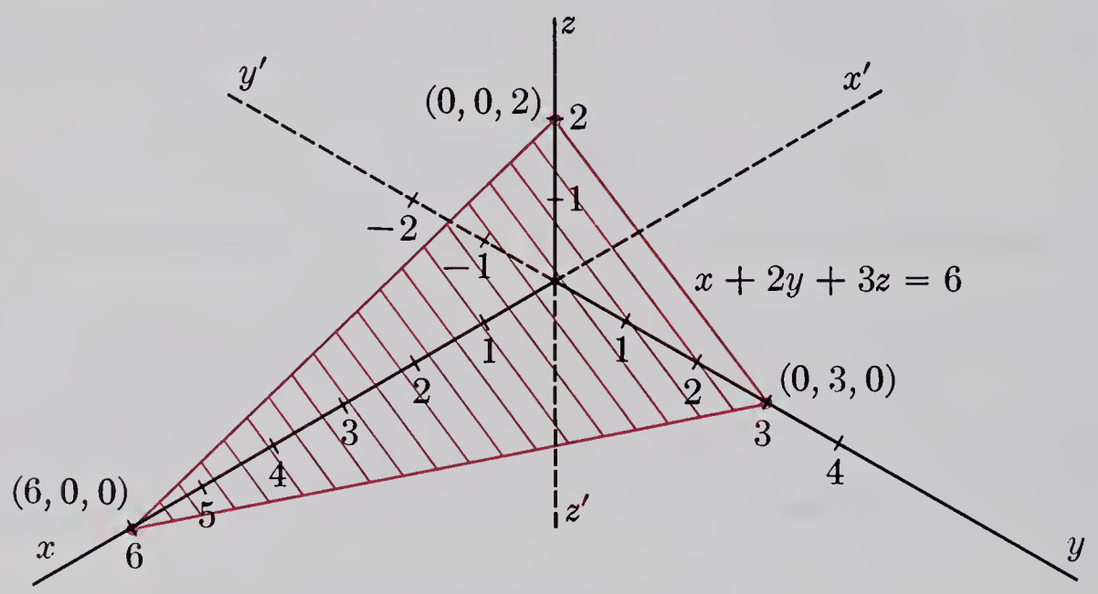


Figure 5.7

These three points cannot be collinear (why not?), and the triangle joining them may be used to represent the plane of the given equation (Figure 5.7).

Example 2. Draw the graph of the equation $x + 2y + 3z = 0$.

Solution: Note that here the constant term is zero; this means that $(0, 0, 0)$ is a point of the locus; that is, the plane passes through the origin. In this case, there is only one point of intersection with the axes. Here we need to find two additional points. These may be determined by assigning values to, say, y and z, and solving for x.

(i) Let $y = -3$, $z = 2$; then $x = 0$.

(ii) Let $y = 0$, $z = 2$; then $x = -6$.

Then $(0, -3, 2)$ and $(-6, 0, 2)$ together with $(0, 0, 0)$ are three points which determine the plane (Figure 5.8).

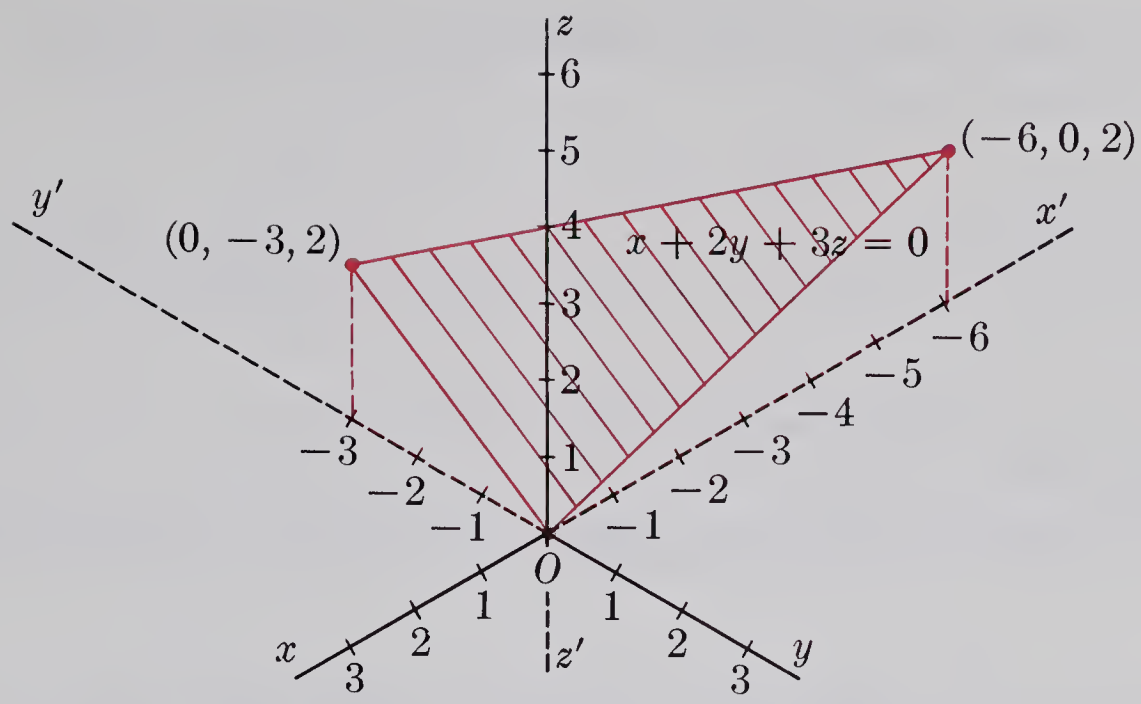


Figure 5.8

Example 3. Sketch the plane whose equation is $x + 2y = 4$.

Solution: Note that this equation does not contain a term in z (or contains the term $0z$ if you prefer); hence, z is unrestricted in value, and if (x_1, y_1) is a number pair such that $x_1 + 2y_1 = 4$, then (x_1, y_1, z), *for any* z, will represent a point on the required plane. The plane is parallel to the z-axis and intersects the xy-co-ordinate plane in the line $x + 2y = 4,\ z = 0$ (Figure 5.9).

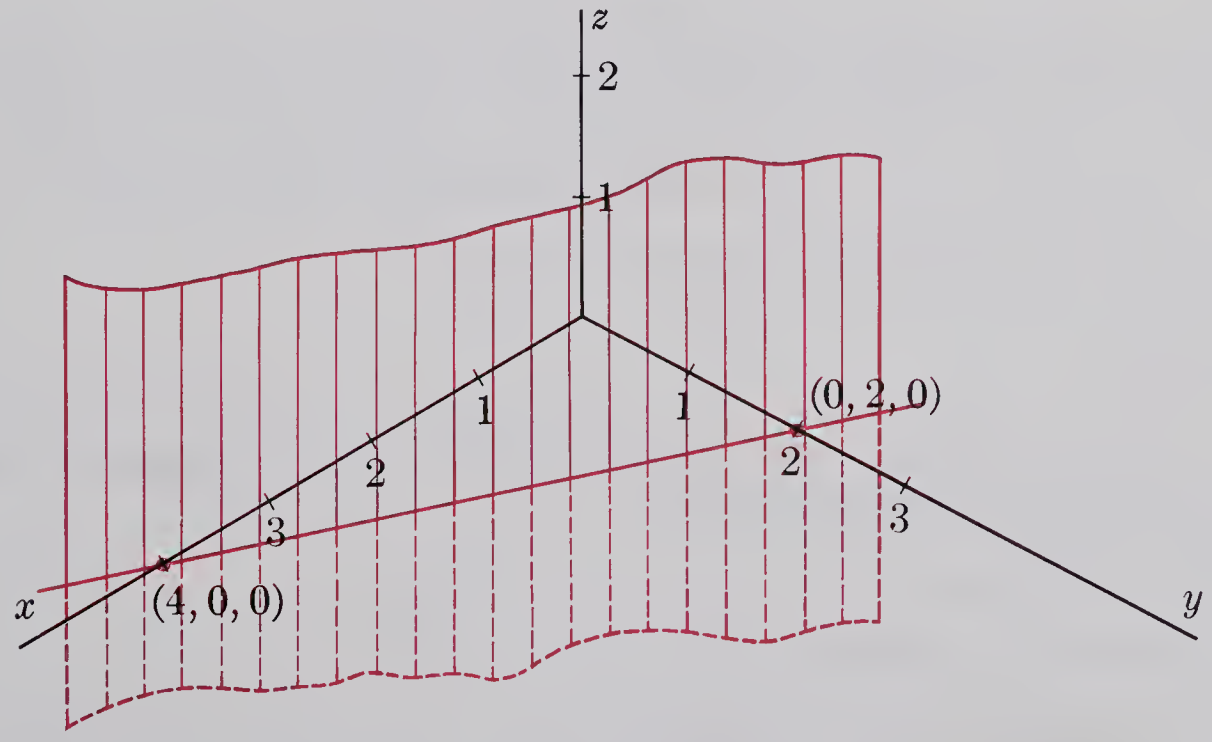


Figure 5.9

EXERCISE 5.4

1. Draw triangles representing the graphs of the following equations.

(a) $4x + y + 3z = 12$	(b) $3x - y + z = 6$
(c) $2x - y - 2z - 4 = 0$	(d) $x = 5$
(e) $3z - 5 = 0$	(f) $6x + y - z - 6 = 0$
(g) $3x - 2y - z + 6 = 0$	(h) $x + y = 4$
(i) $y - z = 6$	(j) $x + 2z - 8 = 0$

2. Use vectors to prove that the plane whose equation is $x + 2y = 4$ is parallel to the z-axis.

3. Prove that the plane whose equation is $by + cz = d$ is parallel to the x-axis.

5.5. Systems of One and Two Linear Equations in x, y, and z

In the next chapter, we shall consider the problem of finding the solution of a system of linear equations. In Sections 5.5, 5.6, and 5.7, we merely point out the possible representations in 3-space of systems of linear equations of the form $ax + by + cz = d$ and their solution sets.

One Equation

We have already seen that a single equation of the form

$$ax + by + cz = d$$

represents a plane in 3-space perpendicular to the vector (a, b, c). The solution set of the single equation is the infinite set of triples (x_1, y_1, z_1) of real numbers such that

$$ax_1 + by_1 + cz_1 = d\ ;$$

these are the co-ordinates of the points in the plane.

Two Equations

A system of two linear equations in x, y, and z representing planes may have the following representations in 3-space.

(i) a single plane

(ii) two distinct parallel planes

(iii) two distinct intersecting planes

These three possibilities are illustrated in Figure 5.10.

In the first situation, the equations must have the form

$$ax + by + cz = d$$

and

$$kax + kby + kcz = kd\,, \qquad k \neq 0\,.$$

These equations are equivalent; the solution set consists of an infinite number of triples (x_1, y_1, z_1) of real numbers such that

$$ax_1 + by_1 + cz_1 = d\,.$$

In the second situation, the planes would both be perpendicular to the same vector (the normal axis to both planes), and so would have equations of the form

$$\begin{aligned} ax + by + cz &= d\,, \\ kax + kby + kcz &= e\,, \qquad \text{where } k \neq 0 \text{ and } e \neq kd\,. \end{aligned}$$

In this case, the solution set is the null set, since any triple (x_1, y_1, z_1) of real numbers such that

$$ax_1 + by_1 + cz_1 = d$$

would be such that

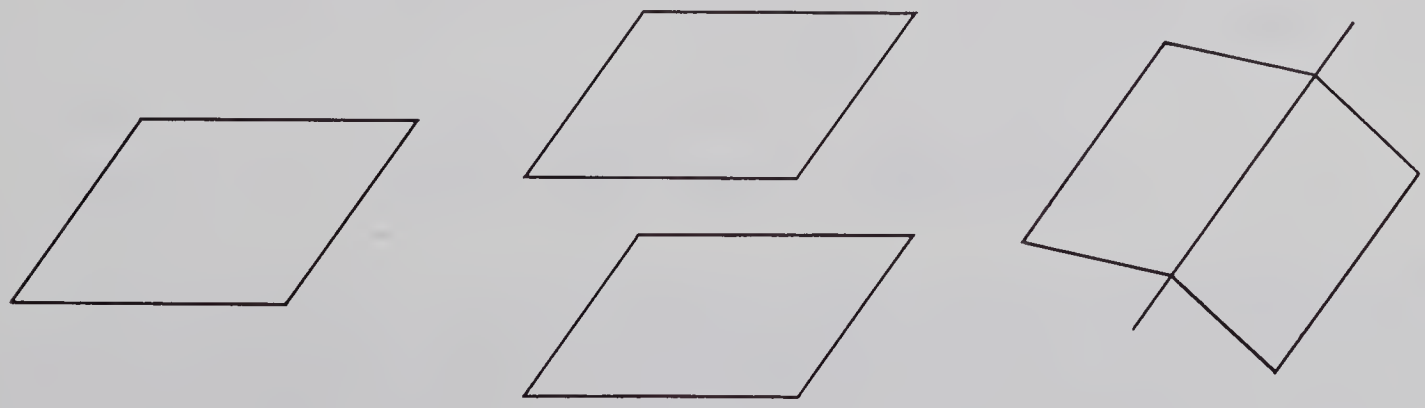
$$kax_1 + kby_1 + kcz_1 = kd \neq e$$

and so would not satisfy the second equation.

In the third situation, the planes would not be perpendicular to the same vector, and so would have equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1\,, \\ a_2x + b_2y + c_2z &= d_2\,, \end{aligned}$$

with $(a_1, b_1, c_1) \neq k\,(a_2, b_2, c_2)$. In this case, the two planes intersect in a line, and so the solution set of the system of two equations is the infinite set of triples (x_1, y_1, z_1) of real numbers representing points on this line of intersection.



case I *case* II *case* III

Figure 5.10

Example 1. Describe the solution sets of the following systems of equations.

(a) $x - 2y + z = 3$
$4x - 8y + 4z = 12$

(b) $x - 2y + z = 3$
$x - 2y + z = 8$

(c) $x - 2y + z = 3$
$x - 2y + 2z = 3$

Solution:

(a) The two equations are equivalent; their solution set is the infinite set of triples (x_1, y_1, z_1) of real numbers such that $x_1 - 2y_1 + z_1 = 3$. These are the co-ordinates of the points on the plane whose equation is $x - 2y + z = 3$.

(b) The two equations represent distinct planes perpendicular to the vector $(1, -2, 1)$; these planes are parallel and do not intersect. The solution set of the system of equations is the null set.

(c) The equations represent planes perpendicular to the vectors $(1, -2, 1)$ and $(1, -2, 2)$, respectively; these vectors are not collinear, and so the planes are not parallel. The planes intersect in a line and the solution set of the system of equations is the infinite set of triples (x_1, y_1, z_1) representing points on that line. These triples satisfy the two equations simultaneously; thus,

$$x_1 - 2y_1 + z_1 = 3\,,$$
$$x_1 - 2y_1 + 2z_1 = 3\,.$$

The result in the last part of Example 1 indicates another way of representing the equations of a line in space. The line l may be represented algebraically by a pair of linear equations representing planes intersecting in that line.

Example 2. Find a system of two linear equations representing planes intersecting in the line

$$\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-4}\,.$$

Solution: We may write these equations as

$$\frac{x-2}{3} = \frac{y+1}{2} \quad \text{and} \quad \frac{y+1}{2} = \frac{z-1}{-4}\,,$$

that is, as the system

$$2x - 3y = 7\,,$$
$$2y + z = -1\,.$$

This is only one such system; there will be an infinite number of systems since each line is the intersection of an infinite number of pairs of planes.

EXERCISE 5.5

Sketch systems of planes represented by the following systems of equations (diagrams such as in Figure 5.10), and describe the solution sets of the systems.

1. $x + 3y - 2z = 4$

2. $2x - 3y = 6$

3. $-x - y + 3z = 2$
   $2x + 2y - 6z = -4$

4. $x + 3y = 2$
   $x + 3y = 3$

5. $x + 3y = 2$
   $y + 3z = 2$

6. $x + y - 3z = 2$
   $4x + y - z = -3$

7. $x + y - 2z = 5$
   $2x + 2y - 4z = 5$

8. $x - y + 2z = 1$
   $4x + 2y - z = 0$

9. $x + 2z = 1$
   $2y - z = 0$

10. $x = 5$
    $3y - 2z = 0$

5.6. Systems of Three Linear Equations in x, y, and z

A system of three linear equations in x, y, and z representing planes may have the following geometric representations in 3-space.

(i) a single plane

(ii) two distinct parallel planes

(iii) three distinct parallel planes

(iv) three distinct planes intersecting in pairs in a set of three parallel lines

(v) two distinct planes intersecting in a line

(vi) two parallel planes and a third plane intersecting them in a pair of parallel lines

(vii) three distinct planes intersecting in a straight line

(viii) three distinct planes intersecting in a single point

From examining the above situations, we see that the solution sets of systems of three linear equations (geometrically, the sets of points common to three planes) can represent a plane ((i) above), a line ((v) and (vii) above), a point ((viii) above), or the null set ((ii), (iii), (iv), and (vi) above). These possibilities are illustrated in Figure 5.11. (The number of the diagram corresponds to that of the case listed above.)

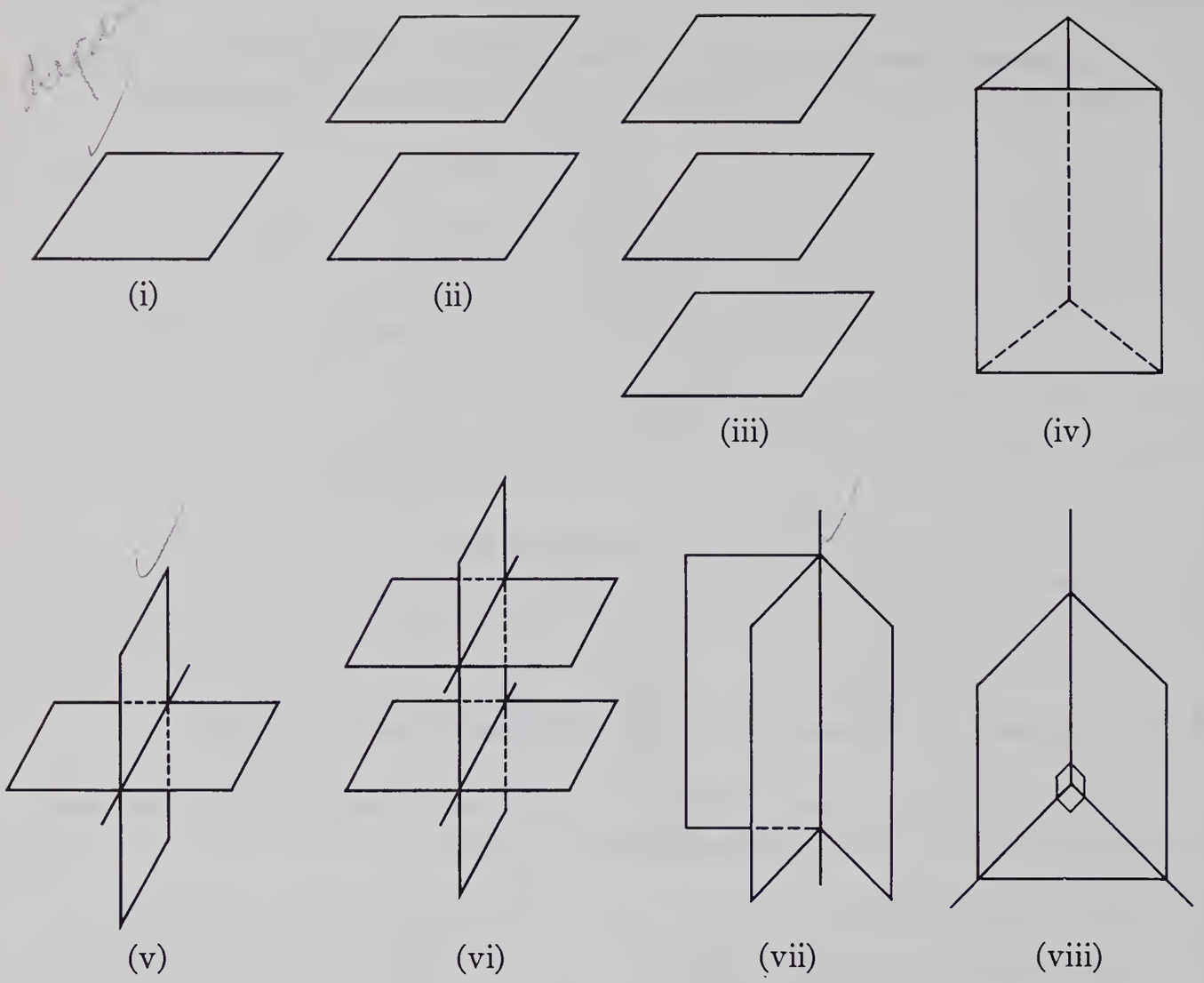


Figure 5.11

Example. Describe the solution sets of the following systems of equations.

(a) $-x + y + 3z = 2$
$2x - 2y - 6z = -4$
$-3x + 3y + 9z = 6$

(b) $-x + y + 3z = 2$
$2x - 2y - 6z = -4$
$-x + y + 3z = 4$

(c) $-x + y + 3z = 2$
$-x + y + 3z = 4$
$2x - 2y - 6z = 10$

(d) $x = 0$
$y = 0$
$x + y = 4$

(e) $-x + y + 3z = 2$
$2x - 2y - 6z = -4$
$x - 3y + 5z = 6$

(f) $-x + y + 3z = 2$
$-x + y + 3z = 4$
$x - 3y + 5z = 6$

(g) $-x + y + 3z = 2$
$x - 3y + 5z = 6$
$-2y + 8z = 8$

(h) $x = 0$
$y = 0$
$z = 0$

Solution:

(a) The three equations are equivalent equations; they each represent the same plane. The solution set consists of an infinite number of triples of real numbers representing points on that plane. See (i) above.

(b) The first two equations represent the same plane (equivalent equations); the third equation represents a plane parallel to the first. The solution set is the null set. See (ii) above.

(c) The three equations represent three distinct parallel planes; the solution set is the null set. See (iii) above.

(d) The first equation represents the yz-co-ordinate plane; the second represents the xz-co-ordinate plane; the third equation represents a plane parallel to the z-axis. The three planes meet in pairs in parallel lines; the solution set of the system is the null set. See (iv) above.

(e) The first two equations represent the same plane (equivalent equations); the third equation represents a plane which is not parallel to the first, since $(1, -3, 5) \neq k(-1, 1, 3)$. The two planes intersect in a line; the solution set consists of all triples of real numbers representing points on that line. See (v) above.

(f) The first two equations represent distinct parallel planes; the third represents a plane not parallel to these, which therefore intersects them in two parallel lines. The solution set is the null set. See (vi) above.

(g) The first two equations represent distinct nonparallel planes which intersect in a line l; the third equation is formed by adding corresponding members of the first two equations. Hence, any triple of real numbers that satisfies the first two equations will also satisfy the third. The third equation must then represent a plane through the line of intersection l. The solution set is the infinite set of triples of real numbers representing points on the line l. See (vii) above.

(h) The three equations represent, respectively, the yz-, xz-, and xy-co--ordinate planes. These planes intersect in a single point, the origin. The solution set of the system of equations consists of the single point $(0, 0, 0)$. See (viii) above.

EXERCISE 5.6

Sketch the systems of planes represented by the following systems of equations (diagrams such as in Figure 5.11), and describe the solution sets of the systems.

1. $\begin{aligned} -x + y - 2z &= 0 \\ x - y + 2z &= 0 \\ 3x - y + 4z &= -3 \end{aligned}$

2. $\begin{aligned} x - 3y + 2z &= 2 \\ 3x - 9y + 6z &= 6 \\ -2x + 6y - 4z &= -4 \end{aligned}$

3. $x - 3y + 2z = 2$
   $3x - 2y + z = -1$
   $-2x + 6y - 4z = -4$

4. $x - 3y + 2z = 2$
   $-2x + 6y - 4z = 4$
   $3x - 2y + z = -1$

5. $4x - y - z = 2$
   $2x + y + 2z = 4$
   $4x + 2y + 4z = 8$

6. $y = 0$
   $z = 0$
   $y + z = 4$

7. $2x + y + z = 5$
   $3x + 2y + z = 7$
   $5x + 3y + 2z = 12$

8. $x = 3$
   $y = 4$
   $x + y = 12$

9. $x = 3$
   $y = 4$
   $x + y = 7$

10. $3x - 2y - z = -4$
    $-x - 2y + z = -2$
    $2x + 3y - z = 5$
    (Hint: Try $(1, 2, 3)$ as a solution.)

11. $x - y - z = -1$
    $4x + 2y + z = 11$
    $-3x - 3y - 2z = -12$
    (Hint: Try $(2, 0, 3)$ as a solution)

12. $2x - y - z = -3$
    $x + 2y - z = -1$
    $3x + y - 2z = -4$
    (Hint: Try $(1, 1, 4)$ as a solution.)

5.7. Systems of More Than Three Linear Equations in *x*, *y*, and *z*

We shall not attempt to spell out here all the various possibilities of geometric representations of systems of four, five, etc., linear equations in x, y, and z. Some of the possibilities will be considered in Exercise 5.7.

We emphasize that, if the system is represented geometrically by a set of planes that have no common point, then the system has no solution. The solution set is the null set. If the planes have a single point in common, the system has a unique solution. The only other possibility is that the planes have an infinite number of points in common; the corresponding system of equations has an infinite number of elements in its solution set.

EXERCISE 5.7

1. Use diagrams to indicate the planes represented by systems of four linear equations in x, y, and z if the systems have
   (a) no solution,
   (b) a unique solution,
   (c) an infinite number of solutions.

   List as many situations as possible. (Hint: four identical planes for (c), two pairs of equivalent parallel planes in (a), etc.)

2. Write a system of equations corresponding to each diagram in question (1).

Chapter Summary

Vector equation of a plane · Parametric equations of a plane (from the vector equation) · The linear equation of the plane · Sketching planes in 3-space Systems of linear equations in x, y, and z

REVIEW EXERCISE 5

1. If noncollinear points A, B, and C have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, locate points P with the following position vectors (use a diagram).
   (a) $\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{1}{3}\mathbf{r}_2 + \frac{1}{3}\mathbf{r}_3$ (b) $\mathbf{r} = \mathbf{r}_1 - 2\mathbf{r}_2 + 2\mathbf{r}_3$
   (c) $\mathbf{r} = \frac{1}{2}\mathbf{r}_1 + \frac{1}{2}\mathbf{r}_3$ (d) $\mathbf{r} = \frac{4}{3}\mathbf{r}_1 - \mathbf{r}_2 + \frac{2}{3}\mathbf{r}_3$

2. Describe the region with equation
   $$\mathbf{r} = (1 - k - m)\mathbf{r}_1 + k\mathbf{r}_2 + m\mathbf{r}_3$$
   if (a) $0 \leq k \leq 1, \quad 0 \leq m \leq 1, \quad k, m \in Re,$
   (b) $-1 \leq k \leq 0, \quad 0 \leq m \leq 1, \quad k, m \in Re.$

3. Find vector and parametric equations of the planes determined by the following sets of points.
   (a) $A\,(0, 0, 0)$, $B\,(3, 2, -1)$, $C\,(4, -1, 1)$
   (b) $A\,(1, 2, -1)$, $B\,(3, -1, 4)$, $C\,(-2, 1, 5)$
   (c) $A\,(-3, 1, 4)$, $B\,(6, -1, -1)$, $C\,(-3, 5, 0)$
   (d) $A\,(5, -2, 0)$, $B\,(-3, 0, 0)$, $C\,(4, 0, -1)$

4. Describe the location of the planes with the following parametric equations.
   (a) $x = -2, \quad y = 4 + k + m, \quad z = 3 - 2k - m, \quad k, m \in Re$
   (b) $x = -2 + 3k - 4m, \quad y = 0, \quad z = 3 - m - 2k, \quad k, m \in Re$

5. Find the equations of the planes perpendicular to the given vectors $\mathbf{v}$ and passing through the given points Q.
   (a) $\mathbf{v} = (1, 1, 5)$, $Q\,(-2, 1, 3)$ (b) $\mathbf{v} = (4, -1, 0)$, $Q\,(2, 2, 5)$
   (c) $\mathbf{v} = (-1, 3, -1)$, $Q\,(2, -1, 5)$ (d) $\mathbf{v} = (2, 0, 5)$, $Q\,(3, -1, 4)$

6. Find the equations of the planes perpendicular to the following vectors and with the given normal intercepts (two solutions for each).
   (a) $(5, 1, 2)$, 3 (b) $(2, -1, -1)$, $\sqrt{3}$

7. (a) Find the intersection of the plane with equation
   $$x - 3y + 2z = 6$$
   and its normal axis.
   (b) Find the distance from the origin to this plane.
   (c) Sketch the plane in part (a).

8. Find the distance from the point $Q(3, -1, -1)$ to the planes whose equations are

   (a) $3x + y - 4z = 24$,
   (b) $6x - y - 2z - 12 = 0$,
   (c) $x + 3y = 6$,
   (d) $x = 2$.

9. Sketch the planes in question (8).

10. Sketch the systems of planes represented by the following systems of equations, and describe the solution sets of the systems.

   (a) $3x - y + 2z = 6$

   (b) $x - 3y - z = 6$
   $3x - 4y + 6z = 12$

   (c) $x + y - 2z = 4$
   $3x + y - 6z = 12$
   $3x + 3y - 6z = 4$

   (d) $x + y - 2z = 4$
   $3x + y - 6z = 12$
   $3x + 3y - 6z = 12$

   (e) $x = 0$
   $y = 0$
   $x + y = 2$

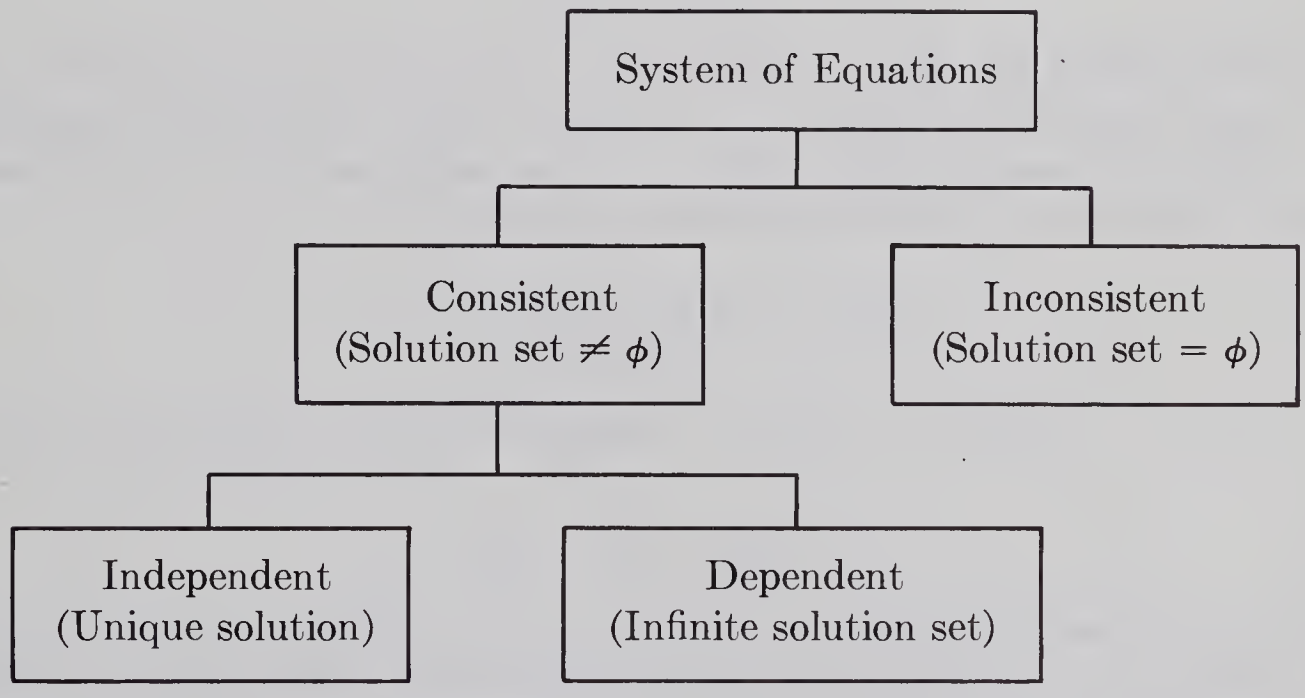
   (f) $y = 3$
   $z = -2$
   $y - z = 5$

Chapter 6

SYSTEMS OF LINEAR EQUATIONS

6.1. Solution of Independent Systems

We shall use the following classification of systems of linear equations according to their solution sets.



We recall that our *modus operandi* in solving an independent system of equations has been to replace the given system by an equivalent system (one that has the same solution set) a number of times until an equivalent system of the form

$$x = a, \quad y = b, \quad z = c, \quad \text{etc.}$$

is obtained. This system then yields the desired solution set. In this section, we shall review this approach for independent systems, and in the next sections, we shall consider some inconsistent and dependent systems. Notice the orderly approach used in the examples.

Example 1. Solve the system of equations

$$2x - 3y = 5\,, \tag{1}$$
$$5x + 2y = -3\,. \tag{2}$$

Solution: Multiply both members of equation (1) by 5 and both members of equation (2) by 2. (We shall indicate this procedure hereafter by writing (1) $\times$ 5 and (2) $\times$ 2.)

(1) $\times$ 5 $\qquad 10x - 15y = 25\,.$ (3)

(2) $\times$ 2 $\qquad 10x + 4y = -6\,.$ (4)

Now the system of equations (3) and (4) is equivalent to the system (1) and (2). So is the system consisting of equation (3) and a new equation, (5), obtained by subtracting the members of equation (4) from the corresponding members of equation (3) (this operation is indicated hereafter by the notation (3) $-$ (4)).

$$10x - 15y = 25\,. \tag{3}$$
$$10x + 4y = -6\,. \tag{4}$$

(3) $-$ (4) $\qquad -19y = 31\,.$ (5)

We call the system (3), (5), an equivalent reduced system of equations. From (5), we may read off the value of y, namely, $-\frac{31}{19}$. Knowing the value of y, we may find the corresponding value of x by substitution in (3). (It is usually simpler to substitute in one of the original equations.)

$$10x - 15(-\tfrac{31}{19}) = 25\,.$$
$$10x = \tfrac{10}{19}\,.$$
$$x = \tfrac{1}{19}\,.$$

Thus

$$x = \tfrac{1}{19}\,, \quad y = -\tfrac{31}{19}$$

is a system of equations equivalent to the original one and its solution is obvious. These values may be checked by substitution in one of the original equations, usually one from which a variable has been eliminated. Here, we choose equation (2).

$$5(\tfrac{1}{19}) + 2(-\tfrac{31}{19}) = -\tfrac{57}{19} = -3\,.$$

The solution set consists of the single ordered pair $\{(\frac{1}{19}, -\frac{31}{19})\}$.

Example 2. Solve the system of equations

$$2x + 5y - 3z = 2\,, \tag{1}$$
$$3x - 2y + 4z = 1\,, \tag{2}$$
$$-5x - 3y + 2z = 3\,. \tag{3}$$

Solution:

Step 1

$$2x + 5y - 3z = 2\,. \qquad (1)$$

(2) $\times 2$ $\qquad 6x - 4y + 8z = 2\,. \qquad (4)$

(3) $\times 2$ $\qquad -10x - 6y + 4z = 6\,. \qquad (5)$

Step 2

$$2x + 5y - 3z = 2\,. \qquad (1)$$

(4) $- 3 \times$ (1) $\qquad -19y + 17z = -4\,. \qquad (6)$

(5) $+ 5 \times$ (1) $\qquad 19y - 11z = 16\,. \qquad (7)$

Step 3

$$2x + 5y - 3z = 2\,. \qquad (1)$$

$$-19y + 17z = -4\,. \qquad (6)$$

(6) + (7) $\qquad 6z = 12\,. \qquad (8)$

Note that we have now reached an equivalent reduced system (1), (6), (8). From (8),

$$z = 2\,.$$

From (6),

$$-19y + 17\,(2) = -4\,.$$

$$y = 2\,.$$

From (1),

$$2x + 5\,(2) - 3\,(2) = 2\,.$$

$$x = -1\,.$$

Then

$$x = -1, \quad y = 2, \quad z = 2$$

is a system equivalent to the original and its solution is obvious. We may check by substitution in equation (2)

$$3\,(-1) - 2\,(2) + 4\,(2) = 1\,.$$

The solution set consists of a single ordered triple $\{(-1, 2, 2)\}$.

EXERCISE 6.1

Use the method of this section to replace each of the following systems by an equivalent reduced system, and use the latter system to obtain the solution of the original system.

1. $x + y = 8$
   $2x + y = 1$

2. $2x - 3y = -4$
   $2x + y = 4$

3. $x + 2 = y$
   $2x + y = 11$

4. $3x - 5y = -7$
   $2x - 5y = -8$

5. $4x - 5y = -2$
   $2x + 7y = 14$

6. $6x + 7y = 27$
   $5x + 2y = 11$

7. $2x + 5y = 6$
$5x - 2y = \frac{1}{2}$

8. $y = \frac{x}{2} - 3$
$3x - 2y = 7$

9. $\frac{4}{3}x + \frac{1}{5}y = 7$
$\frac{2}{3}x + \frac{5}{4}y = -\frac{9}{4}$

10. $x + y - z = -5$
$x + y + z = 9$
$x - y + z = -1$

11. $x + 2y + z = 1$
$2x - 3y - z = 6$
$3x + 5y + 4z = 5$

12. $2x - y + z = 5$
$x + 2y + z = 12$
$3x + y - 2z = 1$

13. $4x - 2y + 3z = 16$
$7x - 5y + 4z = 15$
$2x + 3y + 2z = 21$

14. $2a + 6b - c = -7$
$11a - b + 15c = -25$
$5a - 3b + 11c = -13$

15. $l - m + 2n = 5$
$2l + m - 4n = 0$
$3l + 2m + 2n = 3$

16. $\frac{1}{x} + \frac{1}{y} = 4$
$\frac{1}{x} - \frac{1}{y} = 8$

17. $\frac{2}{3x} + \frac{3}{2y} = 5$
$\frac{3}{2x} + \frac{2}{3y} = \frac{35}{6}$

18. $2x + y - z = 0$
$3x + 4y - 2z = 0$
$4x - y + 3z = 0$

19. $\frac{3}{x} + \frac{2}{y} = 6$
$\frac{4}{x} - \frac{1}{y} = -2$

20. $\frac{1}{5x} - \frac{2}{3y} = 6$
$\frac{3}{5x} + \frac{5}{3y} = -1$

21. $3x - y + 2z = 0$
$3x - 4y + z = 0$
$2x - 6y + 3z = 0$

6.2. Inconsistent Systems

In Section 6.1, we considered systems of two equations in two variables and of three equations in three variables. The systems studied there were independent; that is, they had a unique solution. In this section, we shall consider some systems which we will find are inconsistent. Some of these will consist of m equations in n variables with $m \neq n$.

It will be noted that our approach in discussing these systems is exactly the same as that used in Section 6.1. We replace the given system of equations by an equivalent reduced system from which we may read off the unique solution (as in Section 6.1), or observe that there is no solution (the system is inconsistent).

Example 1. Examine the system of equations

$$x + 3y = 7\,, \tag{1}$$
$$x + 3y = -2\,; \tag{2}$$

that is, find the solution set if it exists, or label the system inconsistent if it does not.

Solution: The equivalent reduced system is

$$x + 3y = 7\,, \tag{1}$$
(2) − (1) $$0y = -9\,. \tag{3}$$

Now there is no real number y for which $0y = -9$; this system is inconsistent.

Example 2. Examine the system of equations

$$x + 2y - z = 5\,, \tag{1}$$
$$3x + 9y - z = 8\,, \tag{2}$$
$$2x + 10y + 2z = -2\,. \tag{3}$$

Solution:

$$x + 2y - z = 5\,. \tag{1}$$
(2) $- 3 \times$ (1) $$3y + 2z = -7\,. \tag{4}$$
(3) $- 2 \times$ (1) $$6y + 4z = -12\,. \tag{5}$$

Then (1), (4), (5) is an equivalent system.

$$x + 2y - z = 5\,. \tag{1}$$
$$3y + 2z = 7\,. \tag{4}$$
(5) $- 2 \times$ (4) $$0z = 2\,. \tag{6}$$

Now (1), (4), (6) is a reduced system equivalent to the original; it has no solution since there is no real number z such that $0z = 2$. The original system is inconsistent.

Example 3. Examine the system

$$x + 3y = 1\,, \tag{1}$$
$$3x + y = 5\,, \tag{2}$$
$$5x - y = -3\,. \tag{3}$$

Solution:

$$x + 3y = 1\,. \tag{1}$$
(2) $- 3 \times$ (1) $$-8y = 2\,. \tag{4}$$
(3) $- 5 \times$ (1) $$-16y = -8\,. \tag{5}$$

Then (1), (4), (5) is an equivalent system.

$$x + 3y = 1\,. \tag{1}$$
$$-8y = 2\,. \tag{4}$$
(5) $- 2 \times$ (4) $$0y = -12\,. \tag{6}$$

System (1), (4), (6) is the reduced system equivalent to the original. We can see that this system has no solution; thus the original system is inconsistent.

EXERCISE 6.2

Examine the following systems of equations. Replace each by an equivalent reduced system from which a solution can be read or the system can be recognized as inconsistent.

1. $3x - 2y = 7$
   $6x - 5y = 3$

2. $3x - 2y = 7$
   $6x - 4y = 3$

3. $x + 4y = 6$
   $-2x - 8y = 7$

4. $x + 2y - 3z = -1$
   $3x - y + z = 3$
   $5x + 3y - 5z = 4$

5. $x + 2y - 3z = -1$
   $3x - y + z = 3$
   $2x + y + z = 5$

6. $2x - y + z = 3$
   $3x + 2y - z = 5$
   $x - 4y + 3z = -2$

7. $x - 3y = 6$
   $2x + y = 3$
   $3x - 2y = 8$

8. $x - 3y = 5$
   $2x + y = 3$
   $3x - 2y = 8$

9. $2x + 3y = 5$
   $3x - 2y = -2$
   $-4x + 7y = 6$

10. $x + 3y - 2z = 7$
    $2x - y - z = 3$
    $3x + 2y - 3z = -5$
    $5x - y - 2z = 2$

11. $x + 2y - z = 3$
    $2x + y - 4z = 5$
    $x + 4y - 2z = -2$
    $2x - y - 3z = -3$

12. $x + 2y - z = 3$
    $2x - y + 3z = -1$
    $x - 4y + 2z = -2$
    $2x + 5y = 4$

13. $x + 2y = 4$
    $3x - y = 7$
    $2x - 3y = 3$
    $5x + y = 3$

14. $x + 2y = 4$
    $3x - y = 7$
    $2x - 3y = 3$
    $5x + 3y = 15$

15. $x + y = 7$
    $3x - 2y = 5$
    $2x + 3y = -1$
    $4x - y = 6$

6.3. Dependent Systems

In this section, we shall repeat the technique of Sections 6.1 and 6.2 with consistent systems of equations which we find are dependent. A study of the examples will show the situation that arises with such a system and also the way of indicating the infinite number of solutions possible.

Example 1. Examine the system of equations

$$2x + y = 4\,, \tag{1}$$
$$4x + 2y = 8\,. \tag{2}$$

Solution:

$$2x + y = 4\,. \tag{1}$$

(2) $- 2 \times$ (1)
$$0y = 0\,. \tag{3}$$

This reduced system is equivalent to the original system. Obviously $0y = 0$ is true for all real numbers y. Let y have the real value k; then from (1),

$$2x + k = 4\,.$$
$$x = \frac{4 - k}{2}\,.$$

The system has an infinite number of solutions; the solution set is

$$\left\{\left(\frac{4 - k}{2}\,,\ k\right)\middle|\ k \in Re\right\}.$$

Particular solutions may be obtained by assigning real values to k; such solutions are $(2, 0)$, $(\frac{3}{2}, 1)$, $(1, 2)$, $(\frac{1}{2}, 3)$, etc. The system is a dependent one.

Example 2. Examine the system of equations

$$x + 3y - 2z = 4\,, \tag{1}$$
$$4x - y + z = -1\,, \tag{2}$$
$$3x - 4y + 3z = -5\,. \tag{3}$$

Solution:

$$x + 3y - 2z = 4\,. \tag{1}$$
$(2) - 4 \times (1)$ $$-13y + 9z = -17\,. \tag{4}$$
$(3) - 3 \times (1)$ $$-13y + 9z = -17\,. \tag{5}$$

Then

$$x + 3y - 2z = 4\,. \tag{1}$$
$$-13y + 9z = -17\,. \tag{4}$$
$(5) - (4)$ $$0z = 0\,. \tag{6}$$

(1), (4), (6) is an equivalent reduced system. Again $0z = 0$ is true for all real numbers z; assigning the real value k to z produces, from (4),

$$-13y + 9k = -17\,.$$
$$y = \frac{9k + 17}{13}\,.$$

From (1),

$$x + 3\left(\frac{9k + 17}{13}\right) - 2k = 4\,.$$
$$x = \frac{1 - k}{13}\,.$$

The solution set is

$$\left\{\left(\frac{1 - k}{13}\,,\ \frac{9k + 17}{13}\,,\ k\right)\middle|\ k \in Re\right\}.$$

Specific solutions include $(\frac{1}{13}, \frac{17}{13}, 0)$, $(0, 2, 1)$, $(\frac{2}{13}, \frac{8}{13}, -1)$, etc. The system is a dependent one.

Example 3. Examine the system

$$-9y + 17z = -6\,. \tag{1}$$

Solution: Whenever a real value is assigned to z, (1) determines a corresponding real value for y; the ordered pairs (y, z) are solutions of (1). The solution set is

$$\left\{\left(\frac{6 + 17k}{9}, k\right) \middle| k \in Re\right\}.$$

Specific solutions include $(\frac{2}{3}, 0)$, $(\frac{23}{9}, 1)$, etc. The system is a dependent one.

Example 4. Examine the system

$$2x + y - 3z = 4\,, \tag{1}$$
$$5x - 2y + z = 7\,. \tag{2}$$

Solution:

(1) $\times 5$ $$10x + 5y - 15z = 20\,. \tag{3}$$

(2) $\times 2$ $$10x - 4y + 2z = 14\,. \tag{4}$$

(3), (4) is an equivalent system, as is (3), (5).

$$10x + 5y - 15z = 20\,. \tag{3}$$

(4) $-$ (3) $$-9y + 17z = -6\,. \tag{5}$$

We can go no further in eliminating variables from our system of equations, and this is our equivalent reduced system. Thus we look for solutions to this last system. From Example 3, the solution set of equation (5) is

$$\left\{\left(\frac{6 + 17k}{9}, k\right) \middle| k \in Re\right\}.$$

Replacing z by k and y by $\dfrac{6 + 17k}{9}$ in (1) yields

$$2x + \frac{6 + 17k}{9} - 3k = 4\,,$$

or

$$x = \frac{5k + 15}{9}\,.$$

The solution set of the original system is

$$\left\{\left(\frac{5k + 15}{9}, \frac{6 + 17k}{9}, k\right) \middle| k \in Re\right\}.$$

Specific solutions include $(\frac{5}{3}, \frac{2}{3}, 0)$, etc. The system is a dependent one.

Example 5. Examine the system

$$-13y + 11z - 7w = -17\,. \tag{1}$$

Solution: Whenever arbitrary real values k and l are assigned to z and w, (1) determines a corresponding real value for y; the ordered triples (y, z, w) are solutions of (1). The solution set is

$$\left\{\left(\frac{17 + 11k - 7l}{13}, k, l\right) \middle| k, l \in Re\right\}.$$

Specific solutions include $(\frac{17}{13}, 0, 0)$, $(\frac{28}{13}, 1, 0)$, $(\frac{10}{13}, 0, 1)$, etc. The system is a dependent one.

Example 6. Examine the system

$$x + 4y - 3z + 2w = 5\,, \tag{1}$$
$$3x - y + 2z - w = -2\,. \tag{2}$$

Solution: The system

$$x + 4y - 3z + 2w = 5\,, \tag{1}$$
(2) $-$ 3 $\times$ (1) $$-13y + 11z - 7w = -17\,, \tag{3}$$

is an equivalent one, and no further elimination of variables is possible; it is a reduced system and we must look for solutions of this system. From Example 5, the solution set of (3) is

$$\left\{\left(\frac{17 + 11k - 7l}{13}, k, l\right) \middle| k, l \in Re\right\}.$$

Replacing w by l, z by k, and y by $\dfrac{17 + 11k - 7l}{13}$ in (1) yields

$$x + 4\left(\frac{17 + 11k - 7l}{13}\right) - 3k + 2l = 5\,.$$

$$x = \frac{-3 - 5k + 2l}{13}\,.$$

The solution set is

$$\left\{\left(\frac{-3 - 5k + 2l}{13}, \frac{17 + 11k - 7l}{13}, k, l\right) \middle| k, l \in Re\right\}.$$

Specific solutions include $(-\frac{3}{13}, \frac{17}{13}, 0, 0)$, $(-\frac{6}{13}, \frac{21}{13}, 1, 1)$, etc. The system is a dependent one.

EXERCISE 6.3

Examine the following systems of equations. State whether they are consistent (independent or dependent) or inconsistent. If a unique solution exists, state the solution. In the case of a dependent system, state the solution set and four specific solutions.

1. $x - 3y = 4$

2. $3x + 2y = -3$

3. $x + 2y - z = 3$

4. $2x - y + 3z = -1$

5. $x + y - z + w = 5$

6. $x - 3y + 2z - w = -2$

7. $x + 2y = 3$
   $2x - y = 2$

8. $x + 2y = 3$
   $3x + 6y = 6$

9. $x + 2y = 3$
   $3x + 6y = 9$

10. $x - 2y - 2z = 6$
    $3x - 4y + z = -1$
    $5x - 8y - 3z = 12$

11. $x - 2y - 2z = 6$
    $3x - 4y + z = -1$
    $5x - 8y - 3z = 11$

12. $x - 2y - 2z = 6$
    $3x - 4y + z = -1$
    $2x - 5y + 3z = -10$

13. $x - 2y + z = 3$
    $2x - y + 3z = 6$

14. $2x + 3y - z = 5$
    $x + 5y - 3z = -2$

15. $3x - 2y + 4z = 2$
    $5x - y - 2z = -3$

16. $x + 2y - z + 2w = 6$
    $3x - y - 2z + 3w = 9$

17. $2x - 3y - z + 5w = 7$
    $3x - 2y + 5z - w = -2$

18. $2x + 3y - z - w = 4$
    $4x + y - 3z + 2w = -1$

19. $x - y + z + 2w - 3t = 2$
    $2x + 3y - z + w - 4t = 5$

Find the general solution and four specific integral solutions (if possible) for the following systems.

20. $2x + 5y = 4$

21. $x + y - 2z = 3$
    $2x - y + 3z = 5$

22. $3x + y - 2z = 4$
    $2x - y + 4z = 2$

6.4. Systems of Homogeneous Equations

The systems that we have been considering thus far have been mainly systems of linear nonhomogeneous equations. A linear homogeneous equation in x and y is one of the form

$$3x + 2y = 0\,;$$

a linear homogeneous equation in x, y, and z is one of the form

$$3x + 2y - z = 0\,.$$

A linear homogeneous equation in certain variables involves only first powers of these variables; the constant term is 0. The equations

$$3x + 2y = 5 \quad \text{and} \quad 3x + 2y - z = -2$$

are linear nonhomogeneous equations.

A system of linear homogeneous equations, such as

(i) $x - 3y = 0,$
$2x + y = 0,$

(ii) $2x + y - z = 0,$
$3x - y + 4z = 0,$

(iii) $x + 2y - z = 0,$
$3x - y + 5z = 0,$
$4x + y + 2z = 0,$

is never inconsistent because (0, 0) is a solution of (i), and (0, 0, 0) is a solution of (ii) and (iii). The only question we are concerned with in such a system is whether it is independent or dependent. This is determined using the technique of the preceding sections.

Example 1. Examine the system

$$x - 3y = 0\,, \tag{1}$$
$$2x + y = 0\,. \tag{2}$$

Solution:

$$x - 3y = 0\,. \tag{1}$$
(2) $- 2 \times$ (1) $$7y = 0\,. \tag{3}$$

The only real value of y for which $7y = 0$ is 0. Replacement of y by this value in (1) yields

$$x - 3(0) = 0\,.$$
$$x = 0\,.$$

Thus $(0, 0)$ is the only solution; the system is independent.

Example 2. Examine the system

$$x - 3y = 0\,, \tag{1}$$
$$-3x + 9y = 0\,. \tag{2}$$

Solution:

$$x - 3y = 0\,. \tag{1}$$
(2) $+ 3 \times$ (1) $$0y = 0\,. \tag{3}$$

Now $0y = 0$ for all real values of y. Replacing y by k in (1) yields

$$x - 3k = 0\,,$$
$$x = 3k\,.$$

The solution set is

$$\{\, (3k, k) \mid k \in Re \,\}\,.$$

Note that this set includes the particular solution $(0, 0)$. The system is dependent.

Example 3. Examine the system

$$x + 3y - 5z = 0\,, \tag{1}$$
$$2x + y - 6z = 0. \tag{2}$$

Solution:

$$x + 3y - 5z = 0\,. \tag{1}$$
$(2) - 2 \times (1)$ $$-5y + 4z = 0\,. \tag{3}$$

(1), (3) is an equivalent reduced system. Now in (3), any real value k may be assigned to z and a corresponding real value obtained for y.

$$-5y + 4k = 0\,.$$
$$y = \tfrac{4}{5}k\,.$$

Replacing z by k and y by $\frac{4}{5}k$ in (1) yields

$$x + 3\left(\frac{4k}{5}\right) - 5k = 0\,.$$
$$x = \tfrac{1}{5}\tfrac{3}{}k\,.$$

The solution set of the original system is

$$\{\, (\tfrac{1}{5}\tfrac{3}{}k, \tfrac{4}{5}k, k) \mid k \in Re \,\}\,.$$

Note that this includes the solution $(0, 0, 0)$. The system is dependent.

Example 4. Examine the system

$$x - 3y + 2z = 0\,, \tag{1}$$
$$3x + y - 4z = 0, \tag{2}$$
$$5x + y - 3z = 0\,. \tag{3}$$

Solution:

$$x - 3y + 2z = 0\,. \tag{1}$$
$(2) - 3 \times (1)$ $$10y - 10z = 0\,. \tag{4}$$
$(3) - 5 \times (1)$ $$16y - 13z = 0\,. \tag{5}$$

(1), (4), (5) is an equivalent system.

$$x - 3y + 2z = 0\,. \tag{1}$$
$\frac{1}{1\,0} \times (4)$ $$y - z = 0\,. \tag{6}$$
$(5) - 16 \times (6)$ $$3z = 0\,. \tag{7}$$

From this equivalent reduced system, we see that the only solution is $(0, 0, 0)$; the original system is independent.

Example 5. Examine the system

$$x - 3y + 2z = 0\,, \tag{1}$$
$$2x - 6y + 4z = 0\,, \tag{2}$$
$$-3x + 9y - 6z = 0\,. \tag{3}$$

Solution:

$$x - 3y + 2z = 0\,. \tag{1}$$
$(2) - 2 \times (1)$ $$0y + 0z = 0\,. \tag{4}$$
$(3) + 3 \times (1)$ $$0z = 0\,. \tag{5}$$

(1), (4), (5) is an equivalent reduced system. From (5), we see that z is an arbitrary real number, say l; from (4), we see that y is also arbitrary, say k. Replacing y by k and z by l in (1) yields

$$x - 3k + 2l = 0\,.$$
$$x = 3k - 2l\,.$$

The solution set is

$$\{\, (3k - 2l, k, l) \mid k, l \in Re \,\}\,.$$

The system is dependent.

EXERCISE 6.4

Examine the following systems of homogeneous equations. Classify them as independent or dependent, and in each case give the solution set.

1. $x - 5y = 0$

2. $3x - 2y + z = 0$

3. $x - 2y - z + 3w = 0$

4. $x + 5y = 0$
   $3x - 2y = 0$

5. $x + 3y = 0$
   $3x + 9y = 0$

6. $x - 2y = 0$
   $3x + y = 0$

7. $x - 2y + z = 0$
   $3x - 6y + z = 0$

8. $x - 2y + z = 0$
   $3x - 6y + 3z = 0$

9. $x - 4y - 3z = 0$
   $3x - y + 2z = 0$

10. $x + y - 3z = 0$
    $3x - y + z = 0$
    $4x + 3y - z = 0$

11. $x + 2y - z = 0$
    $3x - y + z = 0$
    $-x - 9y + 5z = 0$

12. $x + 2y - z = 0$
    $-3x - 6y + 3z = 0$
    $2x + 4y - 2z = 0$

13. $x + 2y - z + 3w = 0$
    $3x - y + 4z - 5w = 0$

14. $2x - y + 3z - w = 0$
    $5x - 2y + 2z + w = 0$

15. $2x - y - z + 2w = 0$
    $6x - 3y - 3z + 6w = 0$

16. $3x + 2y - z + w = 0$
    $6x + 4y - 2z + w = 0$

17. $x - 3y - 2z + w = 0$
    $3x - y + 2z - 2w = 0$
    $4x - 3y + z - 3w = 0$

18. $3x - 2y + z + w = 0$
    $2x + 5y - 3z - w = 0$
    $3x + 4y - z + 4w = 0$

6.5. The Augmented Matrix of a System of Equations. Equivalent Matrices

By a real matrix we shall mean simply a rectangular array of real numbers. Then

(i) $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & \frac{1}{2} & -7 \\ 5 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & \pi & -2 \\ 3 & 1 & \sqrt{2} \\ 4 & 0 & 7 \end{bmatrix}$

(iv) $[3 \quad -5]$ (v) $[1 \quad 0 \quad -4]$ (vi) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

are called matrices. We enclose the rectangular array of real numbers in square brackets. Matrix (i) contains two rows (across) and two columns (up and down) of real numbers. We call it a 2×2 matrix. Matrix (ii) is a 2×3 matrix (two rows, three columns), and (iii) is a 3×3 matrix (three rows, three columns). Matrices (iv) and (v) are also called row vectors, and matrix (vi) is also called a column vector. In a sense, we can consider matrices as extensions of these row and column vectors. At this point, we stress that a matrix is not a number but simply an array of numbers. Matrices and their properties are studied at greater length in Chapter 7.

We may associate two matrices with every system of m linear equations in n variables. First, we take the array of coefficients of the variables; this gives us the matrix of coefficients of the system. For example, associated with the system

$$\begin{aligned} 2x - 3y &= 5\,, \\ 4x + y &= 6\,, \end{aligned}$$

we have

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

as the matrix of coefficents. Now adjoin the constant terms of the system (when written on the right side of the equations) to the coefficients of the variables. The resulting array is called the augmented matrix of the system. For the above system, the augmented matrix is

$$\begin{bmatrix} 2 & -3 & 5 \\ 4 & 1 & 6 \end{bmatrix}.$$

Example 1. Write (a) the matrix of coefficients and (b) the augmented matrix of the system

$$\begin{aligned} 3x - 2y + z &= 5\,, \\ 4x - 2y + 3z &= 0\,. \end{aligned}$$

Solution:

(a) The matrix of coefficients is

$$\begin{bmatrix} 3 & -2 & 1 \\ 4 & -2 & 3 \end{bmatrix}.$$

(b) The augmented matrix is

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 4 & -2 & 3 & 0 \end{bmatrix}.$$

Example 2. Write (a) the matrix of coefficients and (b) the augmented matrix of the system

$$\begin{aligned} 2x - y + z &= 0\,, \\ 4x + 3y - z &= 0\,, \\ x + 2y - 5z &= 0\,. \end{aligned}$$

Solution:

(a) The matrix of coefficients is

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 1 & 2 & -5 \end{bmatrix}.$$

(b) The augmented matrix is

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 4 & 3 & -1 & 0 \\ 1 & 2 & -5 & 0 \end{bmatrix}.$$

DEFINITION. Two matrices are row equivalent if each can be obtained from the other by a sequence of operations of the following types.

Type 1: The interchange of two rows

Type 2: The multiplication of a row by a nonzero real number

Type 3: The addition to the elements of one row of a multiple of the corresponding elements of another row

Note that these are the same types of operations that we performed in changing a given system of equations to an equivalent reduced system.

Example 3. Find the matrices equivalent to

$$\begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix},$$

by

(a) interchanging the two rows,

(b) multiplying row (2) by 5,

(c) adding 3 × row (1) to row (2).

Solution:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \xrightarrow{\text{Interchange rows.}} \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \xrightarrow{\text{Multiply row (2) } \times \text{ (5).}} \begin{bmatrix} 2 & -1 \\ 15 & 25 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \xrightarrow{\text{Add 3 } \times \text{ row (1) to row (2).}} \begin{bmatrix} 2 & -1 \\ 3+6 & 5-3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 9 & 2 \end{bmatrix}$.

Example 4. Find the matrix equivalent to

$$\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$$

by performing *successively* the following operations.

(i) Multiply row (2) by $\frac{1}{2}$.

(ii) Add (-5) $\times$ row (2) to row (1).

(iii) Interchange rows.

Solution:

$$\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{Multiply row (2) } \times \frac{1}{2}.} \begin{bmatrix} 5 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-5) \times \text{ row (2) to row (1).}} \begin{bmatrix} 0 & -11 \\ 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{Interchange rows.}} \begin{bmatrix} 1 & 2 \\ 0 & -11 \end{bmatrix}$$

Note that the matrices $\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & -11 \end{bmatrix}$ are equivalent according to the definition.

EXERCISE 6.5

1. Write the matrix of coefficients and the augmented matrix for each of the systems of equations in Exercise 6.3.

2. (a) Write the augmented matrix of the system

$$3x - 2y = 7,$$
$$5x + 2y = 4.$$

(b) Perform the following operations on the matrix from (a).

(i) Interchange two rows.

(ii) Add 4 × row (1) to row (2).

(iii) Multiply row (1) by 6.

Perform successively the following operations to find a matrix equivalent to the given matrix in questions (3) to (11).

(a) Add 3 × row (2) to row (1).

(b) Interchange rows.

(c) Multiply row (2) by 3.

3. $\begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 1 & -2 \\ 4 & 1 & 6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 & 5 & 0 \\ 2 & 1 & 5 & 2 \end{bmatrix}$

10. $\begin{bmatrix} -3 & 1 & 1 & 2 \\ 0 & 4 & 1 & 5 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 3 & 3 & -4 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

Perform successively the following operations to find a matrix equivalent to the given matrix in question (12) to (14).

(a) Interchange rows (1) and (3).

(b) Multiply row (2) by 2.

(c) Add 3 × row (2) to row (3).

(d) Interchange rows (1) and (2).

12. $\begin{bmatrix} 3 & -1 & 4 \\ 5 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -1 & 6 \\ 1 & 1 & 2 \\ 5 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 4 & 1 & 1 & -3 \\ 0 & 1 & 5 & 2 \end{bmatrix}$

Perform successively the following operations to find a matrix equivalent to the augmented matrix in question (15) to (17).

(a) Add (−1) × row (3) to row (1).

(b) Interchange rows (2) and (3).

(c) Multiply row (2) by 4.

(d) Add $\frac{1}{2}$ × row (1) to row (2).

15. $3x - 2y = 7$
$x + 4y = -1$
$2x - y = 3$

16. $x - 2y = -3$
$3x + y = 5$
$x - y = -1$

17. $x - y = -2$
$2x + y + 3z = 5$
$x + 3y - 2z = -1$

18. $x + 5y = 3$
$-2x + y = -1$
$3x - 2y = 0$

19. $6x - y = 3$
$3x + y = -4$
$2x - 3y = 3$

20. $x + y + 2z = -2$
$3x - y + 3z = 5$
$4x - 2y + 5z = -1$

6.6. Row-Reduced Echelon Forms of a Matrix

Example 1. Perform the following operations successively on the matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & -3 \\ -2 & 4 & 1 \end{bmatrix}.$$

(i) Interchange rows (1) and (2).

(ii) Add $(-3) \times$ row (1) to row (2).

(iii) Add $2 \times$ row (1) to row (3).

Solution:

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & -3 \\ -2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{Interchange rows (1) and (2).}} \begin{bmatrix} 1 & 5 & -3 \\ 3 & -1 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-3) \times \text{ row (1) to row (2).}} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -16 & 11 \\ -2 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Add } 2 \times \text{ row (1) to row (3).}} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -16 & 11 \\ 0 & 14 & -5 \end{bmatrix}$$

Note that in our resulting equivalent matrix, the first entry in row (1) is 1 and the first entries in the other rows are 0.

Example 2. Continue with Example 1 by performing successively the following additional operations.

(i) Multiply row (2) by $-\frac{1}{16}$.

(ii) Add $(-14) \times$ row (2) to row (3).

Solution:

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & -16 & 11 \\ 0 & 14 & -5 \end{bmatrix} \xrightarrow{\text{Multiply row (2) by } -\frac{1}{16}.} \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -\frac{11}{16} \\ 0 & 14 & -5 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-14) \times \text{ row (2) to row (3).}} \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -\frac{11}{16} \\ 0 & 0 & \frac{37}{8} \end{bmatrix}$$

Note that in this matrix, which is equivalent to the original matrix of Example 1, the first nonzero entry in rows (1) and (2) is 1.

Example 3. Continue the above example by multiplying row (3) by $\frac{8}{37}$.

Solution:

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -\frac{11}{16} \\ 0 & 0 & \frac{37}{8} \end{bmatrix} \xrightarrow{\text{Multiply row (3) by } \frac{8}{37}.} \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -\frac{11}{16} \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -\frac{11}{16} \\ 0 & 0 & 1 \end{bmatrix}$$

is equivalent to the original matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & -3 \\ -2 & 4 & 1 \end{bmatrix}$$

and is called a row-reduced echelon form of this matrix.

A row-reduced echelon form of a matrix is an equivalent matrix in which

(i) the first nonzero entry in each row is 1,

(ii) each row contains more initial zero terms than the preceding rows.

The following matrices are in row-reduced echelon form.

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4. Find a row-reduced echelon form for the matrix

$$\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix}.$$

Solution: We may obtain a 1 as the first nonzero entry by multiplying row (1) by $\frac{1}{5}$. We may accomplish the same result, however, by adding $(-1) \times$ row (2) to row (1). In this way we avoid introducing fractions unnecessarily.

$$\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix} \xrightarrow{\text{Add } (-1) \times \text{row (2) to row (1).}} \begin{bmatrix} 1 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-4) \times \text{row (1) to rows (2).}} \begin{bmatrix} 1 & 4 \\ 0 & -13 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2) by } -\frac{1}{13}.} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

Then $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ is a row-reduced echelon form of $\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix}$.

Example 5. Find a row-reduced echelon form for the matrix

$$\begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & 6 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & 6 \end{bmatrix} \xrightarrow{\text{Add } (-1) \times \text{row (1) to row (2).}} \begin{bmatrix} 2 & -1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Interchange rows (1) and (2).}} \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-2) \times \text{row (1) to row (2).}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2) by } -\frac{1}{7}.} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -\frac{3}{7} \end{bmatrix}$$

This is a matrix of the required form.

Example 6. Find a row-reduced echelon form for the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & 3 & -1 & 4 \\ 4 & -1 & 1 & 12 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & 3 & -1 & 4 \\ 4 & -2 & 1 & 12 \end{bmatrix} \xrightarrow{\text{Add } (-2) \times \text{row (1) to row (2).}} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -3 & -2 \\ 4 & -1 & 1 & 12 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-4) \times \text{row (1) to row (3).}} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -3 & -2 \\ 0 & 7 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-1) \times \text{row (2) to row (3).}} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -3 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2) by } \frac{1}{7}.} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{3}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (3) by } \frac{1}{2}.} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{3}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is one of the required form.

It should be pointed out that a row-reduced echelon form of a given matrix is not unique; different sequences of operations may produce different forms. However, these forms will all have the same pattern of initial 1's and 0's in the rows in common. They also will all be equivalent matrices.

Example 7. Find a row-reduced echelon form of the matrix

$$\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix}$$

by using a different sequence of operations from that used in Example 4.

Solution:

$$\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix} \xrightarrow{\text{Multiply row (1) by } \frac{1}{5}.} \begin{bmatrix} 1 & \frac{7}{5} \\ 4 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-4) \times \text{ row (1) to row (2).}} \begin{bmatrix} 1 & \frac{7}{5} \\ 0 & -\frac{13}{5} \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2) by } -\frac{5}{13}.} \begin{bmatrix} 1 & \frac{7}{5} \\ 0 & 1 \end{bmatrix}$$

This is a row-reduced echelon form different from that obtained previously, but the two are equivalent matrices.

Any other row-reduced echelon of the matrix in Example 6 will be of the form

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

EXERCISE 6.6

Find row-reduced echelon forms for the following matrices.

1. $\begin{bmatrix} 1 & 3 \\ 3 & 18 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 5 \end{bmatrix}$ 5. $\begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 7 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & -1 & 6 \\ 5 & 2 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 \\ -1 & 4 \\ 6 & 2 \end{bmatrix}$ 8. $\begin{bmatrix} 3 & 2 \\ -4 & 1 \\ 2 & 3 \end{bmatrix}$ 9. $\begin{bmatrix} -2 & 5 \\ 3 & -4 \\ 3 & 6 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 4 \\ -2 & 5 & 2 \end{bmatrix}$ 11. $\begin{bmatrix} 3 & -2 & 4 \\ 2 & 5 & -1 \\ 4 & -9 & 9 \end{bmatrix}$ 12. $\begin{bmatrix} -2 & 3 & 5 \\ 2 & -3 & -5 \\ 5 & 2 & -1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & 2 & -3 & 1 \\ -2 & 6 & 3 & 0 \end{bmatrix}$ 14. $\begin{bmatrix} 2 & -3 & -3 & 0 \\ 4 & 1 & -5 & 2 \\ -3 & 6 & 1 & -1 \end{bmatrix}$ 15. $\begin{bmatrix} 3 & 3 & 0 & 1 \\ -2 & 4 & 1 & -2 \\ 1 & 7 & 1 & -3 \end{bmatrix}$

Find row-reduced echelon forms for the augmented matrices of the following systems of equations.

16. $2x + 5y = 3$
$3x - y = 7$

17. $3x - 2y = 1$
$x + 4y = -3$
$2x - y = 5$

18. $2x + y - 3z = 1$
$4x - 2y + z = -3$
$-3x + y - 4z = 0$

19. $6x - y = 2$
$2x + y = 1$

20. $3x + 2y = -1$
$2x + 5y = 2$
$-x + 2y = -3$

21. $x + 3y - z = 1$
$4x - y + 3z = 0$
$2x - 2y - 3z = -4$

6.7. Application of the Row-Reduced Echelon Form

In this section, we will see that a row-reduced echelon form of the augmented matrix of a system of equations is the augmented matrix of an equivalent reduced system of equations. Further, we shall see how the solution of a system of equations (or the recognition of inconsistency) can be read from the row-reduced echelon matrix.

Example 1.

(a) Write the augmented matrix of the system

$$3x - 2y = 5\,,$$
$$5x + 3y = 8\,.$$

(b) Find a row-reduced echelon form of the augmented matrix.

(c) Apply the operations used in (b) to the system of equations to produce an equivalent reduced system of equations.

Solution:

(a) The augmented matrix is

$$\begin{bmatrix} 3 & -2 & 5 \\ 5 & 3 & 8 \end{bmatrix}.$$

We perform (b) and (c) in columns.

(b) Augmented matrix		(c) Original System		
	$\begin{bmatrix} 3 & -2 & 5 \\ 5 & 3 & 8 \end{bmatrix}$		$3x - 2y = 5.$	(1)
			$5x + 3y = 8.$	(2)
		Step 1.		
Multiply row (2) $\times$ 3. $\longrightarrow$	$\begin{bmatrix} 3 & -2 & 5 \\ 15 & 9 & 24 \end{bmatrix}$		$3x - 2y = 5.$	(1)
		(2) $\times$ 3	$15x + 9y = 24.$	(3)
		Step 2.		
Add (-5) $\times$ row (1) to row (2). $\longrightarrow$	$\begin{bmatrix} 3 & -2 & 5 \\ 0 & 19 & -1 \end{bmatrix}$		$3x - 2y = 5.$	(1)
		(3) $- 5 \times$ (1)	$0x + 19y = -1.$	(4)
		Step 3.		
Multiply row (1) $\times \frac{1}{3}$ and row (2) $\times \frac{1}{19}$. $\longrightarrow$	$\begin{bmatrix} 1 & -\frac{2}{3} & \frac{5}{3} \\ 0 & 1 & \frac{-1}{19} \end{bmatrix}$	(1) $\times \frac{1}{3}$	$x - \frac{2}{3}y = \frac{5}{3}.$	(5)
		(4) $\times \frac{1}{19}$	$y = \frac{-1}{19}.$	(6)

We have now reached an equivalent reduced system of equations and a row-reduced echelon form of the augmented matrix. The solution of the original system can now be obtained quite easily.

The result of Example 1 should convince us of the truth of the following assertion.

The same operations applied to the augmented matrix of a system of equations to reduce it to row-reduced echelon form will change the system to an equivalent reduced system.

The following example shows how the row-reduced echelon form can be used to read the solution of a system of equations.

Example 2. The following matrices are row-reduced echelon forms of the augmented matrices of certain systems of equations. Write corresponding equivalent reduced systems of equations and examine their solutions.

(a) $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:

(a)

$$x + 4y = -2\,.$$
$$y = 5\,.$$

The solution set is $\{(-22, 5)\}$. The system is independent.

(b)

$$x + 4y = -2\,.$$
$$y = 0\,.$$

The solution set is $\{(-2, 0)\}$. The system is independent.

(c)

$$x + 4y = -2\,.$$
$$0y = 1\,.$$

The solution set is ϕ; the system is inconsistent.

(d)

$$x + 4y = -2\,.$$
$$0y = 0\,.$$

Here y is an arbitrary real number. Let $y = k$, $k \in Re$, and

$$x = -4k - 2\,.$$

The solution set is $\{(-4k - 2, k) \mid k \in Re\}$. The system is dependent.

The next examples combine the above results to show how a system of equations can be examined by using a row-reduced echelon form of the augmented matrix.

Example 3. Find a row-reduced echelon form of the augmented matrix of the system

$$x + 3y - 2z = -4\,,$$
$$3x - y + 4z = 7\,,$$
$$5x - 5y + 10z = 18\,,$$

and use this form to investigate the solution of the system.

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & -1 & 4 & 7 \\ 5 & -5 & 10 & 18 \end{bmatrix}.$$

A row-reduced echelon form for it is found as follows.

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & -1 & 4 & 7 \\ 5 & -5 & 10 & 18 \end{bmatrix} \xrightarrow{\text{Add } (-3) \times \text{row (1) to row (2).}} \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -10 & 10 & 19 \\ 5 & -5 & 10 & 18 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-5) \times \text{row (1) to row (3).}} \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -10 & 10 & 19 \\ 0 & -20 & 20 & 38 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-2) \times \text{row (2) to row (3).}} \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -10 & 10 & 19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2)} \times -\frac{1}{10}.} \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 1 & -1 & -\frac{19}{10} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is a row-reduced echelon form of the augmented matrix; it corresponds to the system of equations

$$\begin{aligned} x + 3y - 2z &= -4\,, \\ y - z &= -\frac{19}{10}\,, \\ 0z &= 0\,. \end{aligned}$$

Then, z is an arbitrary real number.

$$\begin{aligned} z &= k\,. \\ y &= k - \frac{19}{10}\,. \\ x &= -3\left(k - \frac{19}{10}\right) + 2k - 4 = -k + \frac{17}{10}\,. \end{aligned}$$

The solution set is $\left\{\left(-k + \frac{17}{10},\ k - \frac{19}{10},\ k\right) \middle| k \in Re\right\}$. The system is dependent.

Example 4. Examine the system

$$\begin{aligned} x + 2y - z &= 3\,, \\ 4x + 3y - 2z &= -1\,, \\ -x + 3y - z &= 4\,. \end{aligned}$$

Solution: A row-reduced echelon form of the augmented matrix is found as follows.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 3 & -2 & -1 \\ -1 & 3 & -1 & 4 \end{bmatrix} \xrightarrow{\text{Add } (-4) \times \text{row (1) to row (2).}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 2 & -13 \\ -1 & 3 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{Add (1)} \times \text{row (1) to row (3).}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 2 & -13 \\ 0 & 5 & -2 & 7 \end{bmatrix}$$

$$\xrightarrow{\text{Add (1)} \times \text{row (2) to row (3).}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 2 & -13 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2)} \times -\frac{1}{5} \text{ and row (3)} \times -\frac{1}{6}.} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{2}{5} & 1\frac{3}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This row-reduced echelon form corresponds to the equivalent reduced system

$$\begin{aligned} x + 2y - z &= 3\,, \\ y - \tfrac{2}{5}z &= 1\tfrac{3}{5}, \\ 0z &= 1\,, \end{aligned}$$

from which we see that the original system has no solution; the system is inconsistent.

EXERCISE 6.7

Write systems of equations corresponding to the following matrices and examine their solution.

1. $\begin{bmatrix} 1 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 3 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 17. $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 18. $\begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Use the method of Section 6.7 to investigate the following systems of equations.

19. $3x - 2y = 4$
$4x + y = 7$

20. $2x - 5y = 7$
$4x - 10y = 17$

21. $3x - y = 4$
$-6x + 2y = -8$

22. $x + 2y - z = 7$
$3x - y + z = -3$

23. $x - y + 2z = 5$
$3x - y + 3z = 10$

24. $2x - y = 5$
$3x + 2y = -2$
$x - 4y = 12$

25. $x + 3y - 2z = 7$
$2x - y - z = 4$
$5x - 2y + 3z = -3$

26. $x - 2y + z = -2$
$x + 3y = 6$
$3x - y + 2z = 4$

27. $2x + 3y - z = 5$
$4x - 5y + 5z = -3$
$3x - y + 2z = 1$

6.8. Interpretation of Solutions of Systems of Equations

A system of equations in the variables x and y may have no solution (system inconsistent), a unique solution (system independent), or an infinite solution set (system dependent). In the latter case, the general solution is a set of ordered pairs $\{(x, y)\}$ in which each co-ordinate can be expressed in terms of an arbitrary parameter or real number k.

Example 1. Examine the solution set of the system

$$x - 3y = 5\,,$$
$$2x - 6y = 10\,.$$

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & -6 & 10 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & -6 & 10 \end{bmatrix} \xrightarrow{\text{Add } (-2) \times \text{ row (1) to row (2).}} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 0 \end{bmatrix};$$

this row-reduced echelon form corresponds to the system

$$x - 3y = 5\,,$$
$$0y = 0\,.$$

Then y is arbitrary;

$$y = k\,, \qquad k \in Re\,,$$

and

$$x = 3k + 5\,.$$

The solution of the system may be written

$$\{(x, y) \mid x = 3k + 5,\ y = k,\ k \in Re\}\ .$$

The system is dependent.

Now we have seen that the parametric equations

$$\begin{aligned} x &= 3k + 5\,, \\ y &= k\,, \qquad\qquad k \in Re\,, \end{aligned}$$

represent a line in two dimensions (Section 4.2). This confirms what we already know, that is, that the geometric interpretation of the solution set in Example 1 is a line in the plane. The system represents two identical lines.

The following examples concern systems of equations in the variables x, y, and z. Again, the system may have no solution, a unique solution, or an infinite solution set. We examine the parametric equations that arise in the latter case.

Example 2. Examine the system

$$\begin{aligned} x + y + 2z &= -3\,, \\ x + 4y + 7z &= -13\,, \\ 2x - y - z &= 4\,. \end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 4 & 7 & -13 \\ 2 & -1 & -1 & 4 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 4 & 7 & -13 \\ 2 & -1 & -1 & 4 \end{bmatrix} \xrightarrow{\text{Add } (-1) \times \text{row (1) to row (2).}} \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 3 & 5 & -10 \\ 2 & -1 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-2) \times \text{row (1) to row (3).}} \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 3 & 5 & -10 \\ 0 & -3 & -5 & 10 \end{bmatrix}$$

$$\xrightarrow{\text{Add (1) } \times \text{ row (2) to row (3).}} \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 3 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2) } \times \frac{1}{3}.} \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & \frac{5}{3} & -\frac{10}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The row-reduced echelon matrix corresponds to the system

$$\begin{aligned} x + y + 2z &= -3\,, \\ y + \tfrac{5}{3}z &= -\tfrac{10}{3}\,, \\ 0z &= 0\,. \end{aligned}$$

Then z is an arbitrary real number;

$$\begin{aligned} z &= k\,, \qquad k \in Re\,, \\ y &= -\tfrac{5}{3}k - \tfrac{10}{3}\,, \\ x &= \tfrac{5}{3}k + \tfrac{10}{3} - 2k - 3 = -\tfrac{1}{3}k + \tfrac{1}{3}\,. \end{aligned}$$

The solution set is $\{(x, y, z)\}$ with

$$x = -\tfrac{1}{3}k + \tfrac{1}{3}\,, \qquad y = -\tfrac{5}{3}k - \tfrac{10}{3}\,, \qquad z = k\,.$$

We have seen (Section 4.6) that these are the parametric equations of a line in 3-space. This again confirms what we concluded before; the geometric interpretation of the solution set is a line in 3-space. The system represents three planes intersecting in a line.

Example 3. Examine the system

$$\begin{aligned} x + y - 2z &= 5\,, \\ 2x + 2y - 4z &= 10\,. \end{aligned}$$

Solution: The augmented matrix

$$\begin{bmatrix} 1 & 1 & -2 & 5 \\ 2 & 2 & -4 & 10 \end{bmatrix}$$

is equivalent to the row echelon form

$$\begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which corresponds to the system

$$\begin{aligned} x + y - 2z &= 5\,, \\ 0y + 0z &= 0\,. \end{aligned}$$

Here y and z are arbitrary real numbers, $y = k$, $z = l$, and

$$x = -k + 2l + 5\,.$$

The solution set is $\{(x, y, z)\}$ with

$$\begin{aligned} x &= -k + 2l + 5\,, \\ y &= k\,, \\ z &= l\,, \qquad\qquad k, l \in Re\,. \end{aligned}$$

Again we have seen (Section 5.2) that these are the parametric equations of a plane in 3-space. This system of equations represents two planes intersecting in a plane, that is, two identical planes.

EXERCISE 6.8

Examine the systems of equations represented in questions (1) to (27) of Exercise 6.7. Select the dependent systems, write the general solutions in parametric form, and identify the corresponding geometric figure (line or plane).

6.9. Systems of Three Equations in Three Variables

In this section, we will again look at systems of three equations in three variables but from a different point of view. Such a system involves equations of the form

$$ax + by + cz = d\,.$$

We know that this equation is that of a plane in 3-space with normal vector

$$\mathbf{u} = (a, b, c)\,.$$

We have considered the possible geometric interpretations of such systems in Section 5.6. We have seen that such a system is independent only if two of the equations represent planes intersecting in a line (nonparallel planes) and the third equation represents a plane intersecting this line in a point (the line is not in the plane and not parallel to the plane).

Now two planes are parallel if and only if their normal vectors are parallel. Thus,

$$x + 2y + 3z = 6 \quad \text{with normal vector } \mathbf{u} = (1, 2, 3)$$

and

$$2x + 4y + 6z = 7 \quad \text{with normal vector } \mathbf{v} = (2, 4, 6)$$

represent parallel planes, since $\mathbf{v} = 2\mathbf{u}$, whereas

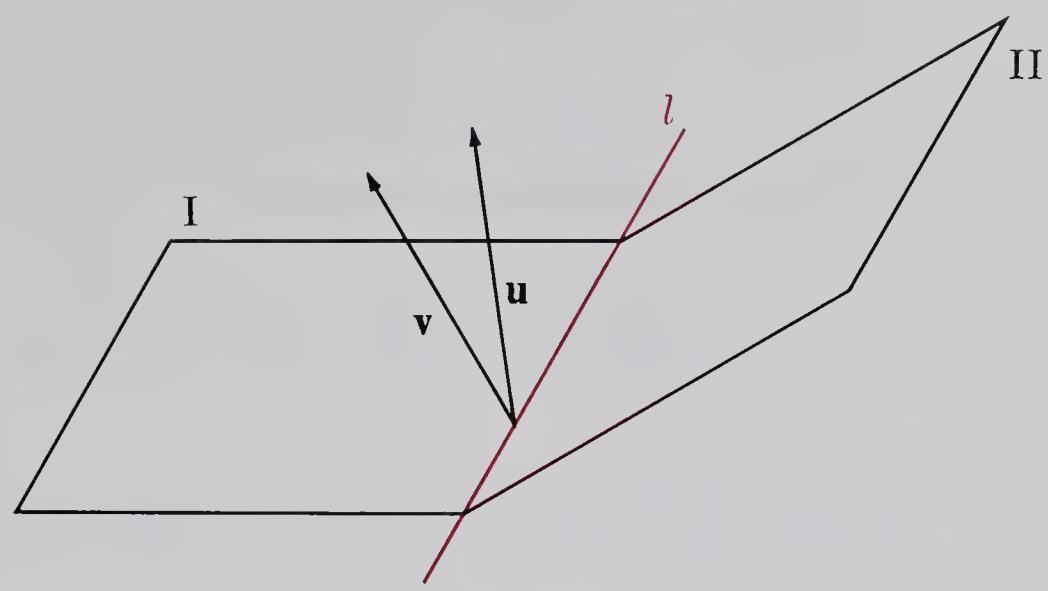
$$x - y + 3z = 2 \quad \text{with normal vector } \mathbf{u} = (1, -1, 3)$$

and

$$2x - 2y + 4z = -3 \quad \text{with normal vector } \mathbf{v} = (2, -2, 4)$$

do not, since $\mathbf{v} \neq k\mathbf{u}$.

The line of intersection l of two nonparallel planes I and II, is perpendicular to the plane determined by the normal vectors $\mathbf{u}$ and $\mathbf{v}$ to the planes.



The condition that a third equation does not represent a plane containing the line l or parallel to the line l is that the normal vector $\mathbf{w}$ of this plane does not lie in the plane determined by $\mathbf{u}$ and $\mathbf{v}$; that is, $\mathbf{w}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Example 1. Use the above considerations to determine whether the following system is independent.

$$\begin{aligned} &(1) && x - 2y + 2z = 5\,. && \mathbf{u} = (1, -2, 2)\,.\\ &(2) && 3x - y + z = -2\,. && \mathbf{v} = (3, -1, 1)\,.\\ &(3) && 4x - 2y + z = 3\,. && \mathbf{w} = (4, -2, 1)\,. \end{aligned}$$

Solution: The normal vectors to the three planes involved are given. Obviously, $\mathbf{u} \neq k\mathbf{v}$, so that (1) and (2) represent nonparallel planes intersecting in a line. The question now is whether we can find k and m such that

$$\mathbf{w} = k\mathbf{u} + m\mathbf{v}\,,$$

that is,

$$\begin{aligned} (4, -2, 1) &= k(1, -2, 2) + m(3, -1, 1)\\ &= (k + 3m,\ -2k - m,\ 2k + m)\,. \end{aligned}$$

This is possible if and only if

$$\begin{aligned} k + 3m &= 4\,,\\ -2k - m &= -2\,,\\ 2k + m &= 1\,, \end{aligned}$$

has a solution (k, m).

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 3 & 4 \\ -2 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

and this matrix is equivalent to the row-reduced echelon form

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & -1 \end{bmatrix}.$$

An examination of the corresponding system of equations shows that no solution (k, m) is possible. Thus, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are noncoplanar vectors, and the original system is independent.

The student will recognize, of course, that the time spent in determining this fact could have just as profitably been spent in obtaining the solution.

Example 2. Determine whether the following system is independent.

(1)	$x + y + 3z = -2$.	$\mathbf{u} = (1, 1, 3)$.
(2)	$x - 4y - 7z = 6$.	$\mathbf{v} = (1, -4, -7)$.
(3)	$4x - y + 2z = 5$.	$\mathbf{w} = (4, -1, 2)$.

Solution: $\mathbf{u}$ and $\mathbf{v}$ are noncollinear and hence determine a plane. This system is independent if and only if real numbers k and m do not exist such that

$$\mathbf{w} = k\mathbf{u} + m\mathbf{v};$$

that is,

$$\begin{aligned} (4, -1, 2) &= k(1, 1, 3) + m(1, -4, -7) \\ &= (k + m, k - 4m, 3k - 7m), \end{aligned}$$

or

$$\begin{aligned} k + m &= 4, \\ k - 4m &= -1, \\ 3k - 7m &= 2. \end{aligned}$$

The augmented matrix is equivalent to

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding system has solution $(3, 1)$; therefore

$$\mathbf{w} = 3\mathbf{u} + \mathbf{v}$$

and so the original system is not independent.

Again, this method tells us nothing further about the solution, that is, whether the system is inconsistent or dependent, and does not give us the solution in the latter case.

Example 3. Determine whether the following system is independent.

(1)	$x - y - 2z = 3$.	$\mathbf{u} = (1, -1, -2)$.
(2)	$5x - 2y + 3z = -4$.	$\mathbf{v} = (5, -2, 3)$.
(3)	$3x - 3y - 6z = 7$.	$\mathbf{w} = (3, -3, -6)$.

Solution: Since $\mathbf{w} = 3\mathbf{u}$, we see that the planes represented by (1) and (3) are parallel planes and so the system of equations is not independent.

EXERCISE 6.9

Use the technique of Section 6.9 to determine if the following systems are independent.

1. $x - y - 2z = 7$
   $3x + 2y - z = -3$
   $x - 2y + z = 5$

2. $2x - 3y + z = 0$
   $5x - y + 3z = -2$
   $x - 2y - z = 5$

3. $3x - y - z = -3$
   $2x + y - 2z = 5$
   $7x + y - 5z = 6$

4. $x - y + 3z = -2$
   $2x - y + z = 5$
   $3x - y - z = 12$

5. $3x - y - z = 2$
   $x - 2y + 3z = -1$
   $2x + y - 4z = 5$

6. $x - y + 2z = -1$
   $2x - 2y + 4z = 3$
   $3x - 3y + 6z = 0$

7. $3x - y - 2z = 5$
   $x + y + 2z = -2$
   $x - 3y - 6z = 7$

8. $2x - 3y - z = 3$
   $x - y + z = 5$
   $-4x + 6y + 2z = 7$

9. $x - 3y - z = 2$
   $2x + y - z = 0$
   $3x - y + 2z = 2$

10. $2x - y - z = 3$
    $x - 2y - 2z = -1$
    $3x - 3y - 3z = 0$

Chapter Summary

Solution of systems of equations · consistent (independent and dependent) and inconsistent systems · equivalent reduced systems
Systems of homogenous equations
The matrix of coefficients and the augmented matrix of a system of equations · equivalent matrices
Row-reduced echelon form and application to the solution of a system of equations
Interpretation of solutions of systems of equations

REVIEW EXERCISE 6

1. Replace each of the following systems by an equivalent reduced system and use the latter to obtain the solution (if existent) of the original system.

(a) $x + y = 7$
$3x - y = 2$

(b) $5x - 2y = 7$
$3x + 4y = -3$

(c) $3x - 2y = 7$
$6x - 4y = 21$

(d) $x - 3y = -3$
$-2x + 6y = 6$

(e) $2x + 3y = 10$
$x - 4y = 7$

(f) $x + 2y = 6$
$3x + 6y = 12$

(g) $x + 3y - 2z = 6$
$2x - y - z = -3$
$3x + 2y - 4z = 10$

(h) $x - 2y + z = -4$
$3x - 4y + 2z = 3$
$4x - 6y + 3z = 6$

(i) $6x - y - z = -2$
$2x + 3y + z = 4$
$2x - 7y - 3z = -10$

(j) $3x + y - 2z = 3$
$x + 4y = -1$
$3y - 2z = 6$

2. Examine the following systems of equations. State whether they are consistent (independent or dependent) or inconsistent. If a solution exists, state the solution set.

(a) $6x - y = 7$

(b) $x - 3y + 2z = 6$

(c) $6x - y + 2z = 7$
$2x + 3y - z = 3$

(d) $3x + 2y - 4z = 12$
$2x - y + 3z = 3$

(e) $x - 2y - z = 4$
$2x - 4y - 2z = 7$

(f) $3x - y - z = -2$
$-9x + 3y - 2z = 6$

(g) $x + 2y - z - 3w = 4$
$2x - y - z + 2w = 7$

(h) $3x - y - 2z + 5w = -2$
$6x - 2y + 3z - w = 5$

3. Examine the following systems of equations. Classify them as independent or dependent; in each case, give the solution set.

(a) $3x + 4y = 0$

(b) $x + 3y + 3z = 0$

(c) $2x - 3y = 0$
$x + 4y = 0$

(d) $3x - 4y = 0$
$-9x + 12y = 0$

(e) $3x - 2y - z = 0$
$2x + 4y - z = 0$

(f) $x + 2y - 6z = 0$
$2x + 4y + 3z = 0$

(g) $x + 3y - 2z = 0$
$3x - y + 4z = 0$
$x + 6y - z = 0$

(h) $3x - y - 2z = 0$
$x + 2y - 4z = 0$
$x - 5y + 6z = 0$

(i) $x + 3y = 0$
$2x - y = 0$
$3x + y = 0$

(j) $x - 3y - z = 0$
$2x - y - z = 0$
$-x + 3y - 2z = 0$

(k) $x + y - 3z + 2w = 0$
$3x - y + z - 2w = 0$

(l) $6x - y - z + 2w = 0$
$3x + 2y - z + w = 0$

4. Are any of the systems in question (3) inconsistent? If not, why not?

5. Write the matrices of coefficients and the augmented matrices for the systems in question (1).

6. Find row-reduced echelon forms for the following matrices.

(a) $\begin{bmatrix} 1 & 2 \\ -7 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 \\ -4 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 4 \\ 4 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -5 \\ 3 & -1 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & -2 & 5 \\ 1 & 7 & 3 \\ 5 & 2 & 9 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 2 & 5 \\ -1 & -1 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} 3 & -2 \\ 5 & 2 \\ 1 & 4 \end{bmatrix}$

(h) $\begin{bmatrix} 2 & -1 & 4 \\ 3 & 3 & 1 \\ 1 & 5 & -3 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 1 & 5 \\ 6 & 2 & 1 \\ 3 & -4 & 5 \end{bmatrix}$

(j) $\begin{bmatrix} 3 & -1 & -1 & 2 \\ 4 & 1 & 5 & 3 \\ -1 & 6 & 3 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$

7. Use the row-reduced echelon form of the augmented matrix of each of the following systems of equations to investigate the solution of the system.

(a) $4x - y = 6$
$3x + 2y = -4$

(b) $3x + 2y = 2$
$-6 + 4y = 4$

(c) $-x + 3y = 2$
$3x - 9y = -6$

(d) $x + 3y - 2z = 4$
$3x - y - z = -2$

(e) $x - 2y + z = -1$
$2x - 4y + 3z = 3$

(f) $x - 2y = 6$
$3x - y = 4$
$-2x + 3y = -1$

(g) $x - 3y + z = 1$
$2x - y - z = -4$
$6x - y + 3z = -1$

(h) $x - 2y - 2z = -1$
$4x - y + 3z = 4$
$x + 5y + 9z = 7$

(i) $3x - 2y + z = 0$
$-x + 3y - z = 0$
$4x + 2y = 0$

(j) $2x - y + 3z = 0$
$x + y - 5z = 0$
$3x - y = 7$

8. Select the dependent systems of question (7), write their solutions in parametric form, and identify the corresponding geometric figure (line or plane).

Chapter 7

MATRICES AND LINEAR TRANSFORMATIONS

7.1. Definition of Matrix

We have defined "matrix" in Section 6.5. At that time, we were concerned with matrices only insofar as they were of use to us in formalizing our technique of replacing a system of equations by an equivalent reduced system to find the system's solution set. We repeat our definition of a matrix here, and amplify our preliminary discussion of matrices. In this chapter and the next, we shall be concerned with matrices as mathematical objects; we shall introduce operations of addition and multiplication for matrices and shall consider the properties of these operations. A 2×2 real matrix is a square array of real numbers, such as

$$\begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix}, \quad \begin{bmatrix} 3 & \frac{1}{2} \\ 2 & -4 \end{bmatrix}, \quad \begin{bmatrix} \pi & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this text, we enclose the array in square brackets; other authors may use different symbols, such as parentheses or double bars.

A 2×2 real matrix consists of two rows and two columns of real numbers. In general, a 2×2 real matrix has the form

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{columns}} \Big\} \text{rows}$$

in which a_{ij} represents the real number in row i and column j; a_{12} is the entry or component in row 1 and column 2, etc. The two rows are

$$[a_{11} \quad a_{12}] \quad \text{and} \quad [a_{21} \quad a_{22}].$$

The two columns are

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

Example 1. Identify a_{11}, a_{12}, a_{21}, and a_{22} in the matrix

$$\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}.$$

Solution:

$$a_{11} = 3; \quad a_{12} = -1; \quad a_{21} = 0; \quad a_{22} = 5.$$

A capital letter, A, B, C, etc., is often used to indicate a certain matrix; also, the matrix in which the element in row i and column j is a_{ij} is represented by the symbol $[a_{ij}]$.

An $m \times n$ real matrix is a rectangular array consisting of m rows and n columns of real numbers. (Such a matrix is said to have dimensions $m \times n$.)

$$\begin{bmatrix} 3 & 0 & -1 \\ 5 & 2 & 0 \end{bmatrix} \quad \text{is a } 2 \times 3 \text{ real matrix.}$$

$$\begin{bmatrix} 5 & -1 \\ 4 & \frac{1}{2} \\ -7 & 0 \end{bmatrix} \quad \text{is a } 3 \times 2 \text{ real matrix.}$$

$$\begin{bmatrix} 5 & 2 & -1 \\ 3 & 1 & 5 \\ 6 & 2 & 4 \end{bmatrix} \quad \text{is a } 3 \times 3 \text{ real matrix.}$$

$$[1 \quad 5] \quad \text{is a } 1 \times 2 \text{ real matrix.}$$

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \text{is a } 2 \times 1 \text{ real matrix.}$$

Note that vectors such as $(3, 5)$ and $(-1, 2, 4)$ may be considered as 1×2 and 1×3 real matrices, $[3 \quad 5]$ and $[-1 \quad 2 \quad 4]$. We refer to these vectors as row vectors; 2×1 and 3×1 real matrices such as

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

are called column vectors.

The components a_{ij} of a matrix are not necessarily real numbers. However, we shall restrict our attention in this course to real matrices, that is, matrices in which the components are real numbers.

In Chapter 2, we defined equality of algebraic vectors. Two vectors are equal if and only if they have the same dimension (same number of components) and corresponding components are equal. We extend this idea to the definition of equality of matrices.

DEFINITION. Two matrices are equal if and only if

(i) they have the same dimensions (same number of rows and same number of columns)

and

(ii) corresponding components are equal.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{iff} \quad \begin{cases} a_{11} = b_{11}, & a_{12} = b_{12}, \\ a_{21} = b_{21}, & a_{22} = b_{22}. \end{cases}$$

Example 4. Can $A = B$ or $B = C$ or $C = A$ if

$$A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} x & -1 \\ 5 & y \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 & 0 \\ 5 & 2 & 0 \end{bmatrix}?$$

Solution:

(i) $A = B$ iff $x = 3$ and $y = 2$.

(ii) $B \neq C$ since B and C do not have the same dimensions.

(iii) $C \neq A$ since C and A do not have the same dimensions.

EXERCISE 7.1

1. List a_{11}, a_{13}, and a_{22} (if they exist) for the following matrices.

(a) $\begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -2 & 0 \\ 1 & 5 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & 3 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 5 \\ -1 \\ 3 \\ 6 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 0 & -1 \\ -3 & 5 & 0 \\ 0 & -2 & 4 \end{bmatrix}$

2. State the dimensions of the matrices in question (1). Are there any row or column vectors in question (1)?

3. Are any of the matrices in question (1) equal? Explain.

4. Under what conditions can $A = B$ if

$$A = \begin{bmatrix} 2 & a \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 \\ b & 3 \end{bmatrix}?$$

5. Under what conditions can $A = B$ if

$$A = \begin{bmatrix} 5 & -3 \\ b & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 3 \\ 5 & -a \end{bmatrix}?$$

6. If $A = [a_{ij}] = \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix}$ and $B = [b_{ij}] = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}$, find

(a) $a_{11}b_{11} + a_{12}b_{21}$, (b) $a_{11}b_{12} + a_{12}b_{22}$,

(c) $a_{21}b_{11} + a_{22}b_{21}$, (d) $a_{21}b_{12} + a_{22}b_{22}$.

7. Repeat question (6) for the matrices

$$A = [a_{ij}] = \begin{bmatrix} 5 & 0 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad B = [b_{ij}] = \begin{bmatrix} -3 & 2 \\ 2 & 3 \end{bmatrix}.$$

7.2. Addition of Matrices • Multiplication by a Real Number

In Chapter 2, we defined addition of vectors of the same dimension. The definition was in terms of the addition of corresponding components. This definition is now extended to the addition of equidimensional matrices (matrices of the same shape). Again the addition of two such matrices is performed by adding corresponding components.

DEFINITION. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

This definition may be extended to the sum of two $m \times n$ matrices.

Example 1. Find $A + B$ if

$$A = \begin{bmatrix} 3 & -1 \\ 5 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 \\ -3 & 5 \end{bmatrix}.$$

Solution:

$$A + B = \begin{bmatrix} 3 + (-4) & (-1) + 1 \\ 5 + (-3) & 7 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 2 & 12 \end{bmatrix}.$$

Note that under the above definition, the set of all 2×2 real matrices is closed under addition.

In Chapter 2, we also defined the product of a vector and a real number or scalar; again, the definition was in terms of the multiplication of each component by the real number. This definition is extended to the product of a matrix and a real number.

DEFINITION. If k is a real number and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then

$$kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}.$$

Again, this definition may be extended to the product of a real number and an $m \times n$ matrix.

Example 2. Find $7A$ and $(-2)A$ if

$$A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \end{bmatrix}.$$

Solution:

$$7A = \begin{bmatrix} 7 \times (-2) & 7 \times 1 \\ 7 \times 5 & 7 \times 0 \end{bmatrix} = \begin{bmatrix} -14 & 7 \\ 35 & 0 \end{bmatrix}.$$

$$(-2)A = \begin{bmatrix} (-2) \times (-2) & (-2) \times 1 \\ (-2) \times 5 & (-2) \times 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -10 & 0 \end{bmatrix}.$$

EXERCISE 7.2

1. If

$$A = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ -7 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 4 & 4 \\ -2 & 1 \end{bmatrix},$$

find the following.

(a) $A + B$ (b) $B + C$ (c) $C + A$ (d) $B + A$
(e) $C + B$ (f) $A + C$ (g) $A + (B + C)$ (h) $(A + B) + C$

2. Repeat question (1) for

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -7 \\ 5 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -5 & 4 \\ 1 & 6 \end{bmatrix}.$$

3. Find (a) $6A$ (b) $(-3)B$ (c) $\frac{1}{2}C$ for the matrices in question (2).

4. For the matrices in question (2), find
   (a) $3A + 2B$, (b) $(-2)A + 5C$, (c) $(-4)A + 2B + (-3)C$.

5. If
$$A = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -2 \\ 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix},$$
   find the following.
   (a) $A + D$ (b) $A + (B + C)$ (c) $(A + B) + C$ (d) $3A + 2B$
   (e) $[A + 3B] + 2C$ (f) $3A + [(-2)B + D]$
   (g) $[5B + (-1)C] + 3D$ (h) $[A + 2B] + [3C + 4D]$

6. If
$$A = \begin{bmatrix} -1 & 2 \\ 7 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix},$$
   find a, b, c, and d so that
   (a) $A + B = C$, (b) $2A + (-3)B = C$, (c) $5C + (-a)A = 3B$.

7. Explain the statement "The set of all 2×2 real matrices is closed under multiplication by real numbers."

8. Assuming that the definition given for addition of 2×2 real matrices holds for any two equidimensional matrices, find $A + B$, where possible, for the following matrices.
   (a) $A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -1 & 2 \\ 6 & 4 & -3 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 5 & -1 \\ -4 & 6 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 0 & -6 \\ -4 & 5 \end{bmatrix}$
   (c) $A = \begin{bmatrix} 5 & -2 \\ -1 & 3 \\ 6 & 0 \\ 7 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
   (d) $A = \begin{bmatrix} 3 & 4 & -2 \\ -1 & 0 & 5 \\ 2 & 7 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 3 \\ -4 & 5 & -5 \\ 3 & -7 & 1 \end{bmatrix}$
   (e) $A = \begin{bmatrix} -1 & 2 & 3 \\ 3 & 0 & 6 \\ 5 & -4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 0 \\ 4 & -4 & 0 \\ -1 & 6 & 0 \end{bmatrix}$
   (f) $A = \begin{bmatrix} -1 & 2 & 3 \\ 3 & 0 & 6 \\ 5 & -4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & -4 \\ -1 & 6 \end{bmatrix}$

9. Is the set of all (a) 3×3 matrices (b) 2×3 matrices (c) 3×5 matrices closed under addition?

7.3. Properties of Addition of Matrices

The problems of Exercise 7.2 may have indicated some of the properties of addition of 2×2 real matrices. We will illustrate these properties by examples but leave their general proof to the reader.

Example 1. Find $A + B$ and $B + A$ if

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A + B &= \begin{bmatrix} 6 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 6 + (-3) & (-1) + 5 \\ 2 + 0 & 4 + 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 8 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} B + A &= \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} (-3) + 6 & 5 + (-1) \\ 0 + 2 & 4 + 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 8 \end{bmatrix}. \end{aligned}$$

Notice that

$$A + B = B + A$$

for the matrices of this example. This result is true in general. Addition of 2×2 real matrices is commutative.

Example 2. Find $(A + B) + C$ and $A + (B + C)$ if A and B are as in Example 1 and

$$C = \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}.$$

Solution: Using the result from Example 1, we find that

$$\begin{aligned} (A + B) + C &= \begin{bmatrix} 3 & 4 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 0 & 13 \end{bmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} A + (B + C) &= A + \left\{ \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 6 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 0 & 13 \end{bmatrix}. \end{aligned}$$

Notice that

$$(A + B) + C = A + (B + C)$$

for the matrices of this example. This result is also true in general. Addition of 2×2 real matrices is associative.

It is easily seen that the matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

is an additive identity under addition of 2×2 real matrices, since

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

It is left as an exercise (see question (10), Exercise 7.3) for the reader to show that this identity is unique.

Example 3. Find $A + (-1)A$ if

$$A = \begin{bmatrix} 3 & 6 \\ -2 & -5 \end{bmatrix}.$$

Solution:

$$(-1)A = \begin{bmatrix} -3 & -6 \\ 2 & 5 \end{bmatrix}.$$

$$A + (-1)A = \begin{bmatrix} 3 & 6 \\ -2 & -5 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= O.$$

We see that for this matrix A, the matrix $(-1)A$ is the additive inverse; this result is also true in general. We refer to the additive inverse or negative of A as $-A$. The additive inverse of a matrix is unique (see question (11), Exercise 7.3).

Subtraction of matrix B from matrix A is defined as the addition of $-B$ to A.

$$A - B = A + (-B).$$

Example 4. Find $3A - 4B$ if

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -7 \\ 1 & 3 \end{bmatrix}.$$

Solution:

$$3A = \begin{bmatrix} 9 & -3 \\ 3 & 15 \end{bmatrix} \quad \text{and} \quad (-4)B = \begin{bmatrix} 8 & 28 \\ -4 & -12 \end{bmatrix}.$$

$$\begin{aligned} 3A - 4B &= 3A + (-4)B \\ &= \begin{bmatrix} 9 & -3 \\ 3 & 15 \end{bmatrix} + \begin{bmatrix} 8 & 28 \\ -4 & -12 \end{bmatrix} \\ &= \begin{bmatrix} 17 & 25 \\ -1 & 3 \end{bmatrix}. \end{aligned}$$

Example 5. Show that $3(A + B) = 3A + 3B$ if

$$A = \begin{bmatrix} 7 & -1 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -5 & 1 \\ -3 & 2 \end{bmatrix}.$$

Solution:

$$3A = \begin{bmatrix} 21 & -3 \\ 6 & 12 \end{bmatrix} \quad \text{and} \quad 3B = \begin{bmatrix} -15 & 3 \\ -9 & 6 \end{bmatrix}.$$

$$3A + 3B = \begin{bmatrix} 6 & 0 \\ -3 & 18 \end{bmatrix}.$$

$$A + B = \begin{bmatrix} 2 & 0 \\ -1 & 6 \end{bmatrix}.$$

$$3(A + B) = \begin{bmatrix} 6 & 0 \\ -3 & 18 \end{bmatrix}.$$

Therefore, $3(A + B) = 3A + 3B$ for these matrices A and B.

The result obtained in Example 5 is also true in general.

$$k(A + B) = kA + kB\,, \qquad k \in Re\,.$$

EXERCISE 7.3

1. Verify the commutativity of addition of 2×2 real matrices by calculating (a) $A + B$ (b) $B + A$ (c) $A + C$ (d) $C + A$ (e) $B + C$ (f) $C + B$ if

$$A = \begin{bmatrix} 1 & 7 \\ -3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -3 \\ 5 & -2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 4 & -2 \\ 1 & 7 \end{bmatrix}.$$

2. Prove that addition of 2×2 real matrices is commutative.
3. Use the matrices of question (1) to show that
$$(A + B) + C = A + (B + C)\,.$$
4. Prove that addition of 2×2 real matrices is associative.
5. State the negatives of the matrices of question (1).

6. Find
   (a) $3A - 2B + C$ (b) $A - 5B - 2C$ (c) $3A - 4B - C$
   for the matrices A, B, and C of question (1).
7. Check that
   (a) $4(A + B) = 4A + 4B$ (b) $(-2)(B + C) = (-2)B + (-2)C$
   (c) $\frac{1}{2}(A + B) = \frac{1}{2}A + \frac{1}{2}B$ (d) $3(A + B + C) = 3A + 3B + 3C$
   for the matrices of question (1).
8. Prove that $k(A + B) = kA + kB$ if A and B are 2×2 real matrices and k is a real number.
9. Is the set of 2×2 real matrices closed under subtraction?
10. Show that the matrix
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
    is the unique additive identity in the set of 2×2 real matrices.
11. Show that $(-1)A$ is the unique additive inverse of A in the set of 2×2 real matrices.

7.4. Linear Transformations of the Plane

The system of equations

$$\left.\begin{aligned} u &= 3x - 2y \\ v &= 4x + y \end{aligned}\right\} \tag{1}$$

produces a mapping $(x, y) \rightarrow (u, v)$ of the set of ordered pairs (x, y) *into* or *onto* the set of ordered pairs (u, v) in the sense that, whenever a pair of values is assigned to x and y, a pair of values is produced for u and v. (The distinction between *into* and *onto* will be clear later on.)

Geometrically, the system of equations produces a mapping or transformation of the plane in the sense that, to every point P with co-ordinates (x, y), there corresponds a point Q with co-ordinates (u, v). We call Q the *image* of P under this mapping.

If we use a letter such as T to represent this transformation or mapping, we write

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

to indicate that $\begin{bmatrix} x \\ y \end{bmatrix}$ is changed into or mapped onto $\begin{bmatrix} u \\ v \end{bmatrix}$, by the transformation T.

Thus, for the mapping T determined by equations (1),

$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

since

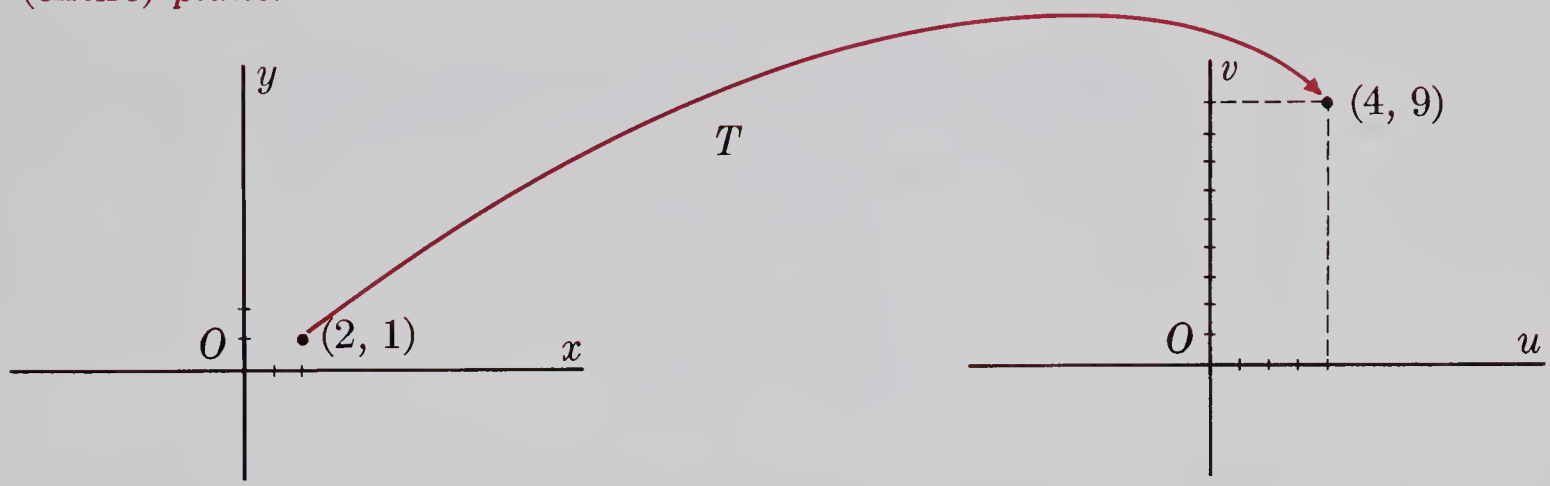
$$u = (3)(2) - (2)(1) = 4\,,$$

and

$$v = (4)(2) + (1)(1) = 9\,.$$

We also write $(2, 1) \to (4, 9)$ to indicate that $(4, 9)$ is the image of $(2, 1)$. The latter notation may seem more useful in order to indicate that ordered pairs are mapped onto ordered pairs or points onto points, but the former notation will prove desirable in our further study of matrices.

It is apparent that every point P with co-ordinates (x, y) will have an image under this transformation T; we say that T is a transformation or mapping *of the* (entire) *plane.*



Example 1. Find the images of the points with the co-ordinates

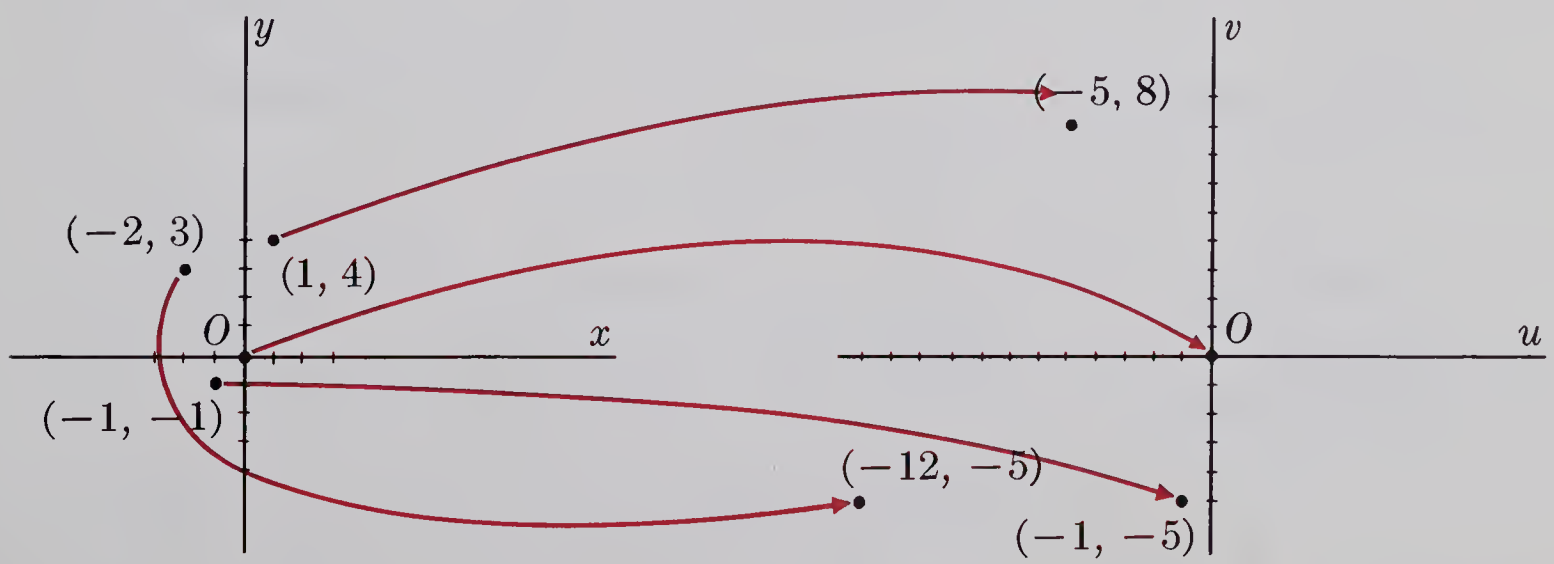
(a) $(0, 0)$ (b) $(1, 4)$ (c) $(-2, 3)$ (d) $(-1, -1)$

under the transformation T determined by equations (1), and show the images on a diagram.

Solution:

(a) $T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ (b) $T\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}-5\\8\end{bmatrix}$

(c) $T\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}-12\\-5\end{bmatrix}$ (d) $T\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-5\end{bmatrix}$



We see that the transformation T determined by equations (1) is a transformation of the entire plane; that is, it provides an image for each point in the xy-plane. Now we ask "Is every point of the uv-plane the image of some point in the xy-plane?" We consider a particular point in the following example.

Example 2. Is the point with co-ordinates $(5, -7)$ the image of some point in the xy-plane under the transformation T determined by equations (1)?

Solution: Rephrasing the question, we are asking if the system of equations

$$5 = 3x - 2y$$
$$-7 = 4x + y$$

has a real solution. Using our techniques for solving such a system, we find the solution set to be $\{(-\frac{9}{11}, -\frac{41}{11})\}$; that is, the point with co-ordinates $(-\frac{9}{11}, -\frac{41}{11})$ has as its image the point with co-ordinates $(5, -7)$ under the mapping T.

$$T \begin{bmatrix} -\frac{9}{11} \\ -\frac{41}{11} \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}.$$

We now see that the question "Is every point of the uv-plane the image of some point under mapping T?" is equivalent to the question "Does the system of equations

$$a = 3x - 2y$$
$$b = 4x + y$$

have a real solution for every pair (a, b) of real numbers?" The answer is "Yes" and its justification is required in question (8). Exercise 7.4. At the same time, we shall see that the solution in each case is unique, so that each point in the uv-plane will be the image of exactly one point in the xy-plane. The mapping T provides a one-to-one correspondence between the points of the xy-plane and the points of the uv-plane. We also say that T is a one-to-one mapping of the xy-plane onto the uv-plane. Not all mappings have this character, as we shall see in the next section.

The basic ingredients of the transformation T are the coefficients of the variables in the system of equations, since

$u = 3x - 2y$	$\lambda = 3\alpha - 2\beta$	$m = 3r - 2s$	$X = 3V - 2W$
$v = 4x + y$	$\delta = 4\alpha + \beta$	$n = 4r + s$	$Y = 4V + W$

are all essentially the same transformation. Thus, the transformation is determined by the coefficients

$$\begin{matrix} 3 & -2 \\ 4 & 1 \end{matrix}.$$

In this way, we may associate a matrix with every such transformation; that is,

$$T \quad \begin{cases} u = 3x - 2y \\ v = 4x + y \end{cases}$$

corresponds to the matrix of coefficients

$$T = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}.$$

We use the same symbol T to refer to both the transformation and the matrix corresponding to the transformation but there should be no confusion.

Example 3. Write the matrix associated with the transformation

$$R \quad \begin{cases} u = x + 3y \\ v = 2x + 4y \end{cases}.$$

Solution: The corresponding matrix is the matrix of coefficients

$$R = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Example 4. Write a linear transformation corresponding to the matrix

$$S = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}.$$

Solution: Such a linear transformation may be written as

$$\begin{aligned} u &= 3x + 4y, \\ v &= -x + 2y, \end{aligned}$$

or as

$$\begin{aligned} m &= 3r + 4s, \\ n &= -r + 2s, \end{aligned}$$

etc.

Finally, we point out that in this chapter we are restricting the examples mainly to transformations corresponding to 2×2 real matrices.

EXERCISE 7.4

1. For the transformation

$$T \quad \begin{cases} u = x + 3y \\ v = 2x - y \end{cases}$$

find the following.

(a) $T\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $T\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (d) $T\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

(e) $T\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (f) $T\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ (g) $T\begin{bmatrix} -1 \\ -5 \end{bmatrix}$ (h) $T\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Show these points and their images on a diagram.

2. For the transformation T in question (1), find the points whose images are the points with the following co-ordinates.

   (a) $(0, 0)$ (b) $(-1, 3)$ (c) $(5, 0)$ (d) $(-2, 4)$

3. State the matrices corresponding to the following transformations.

   (a) R $\begin{cases} u = 3x - y \\ v = 5x + 2y \end{cases}$ (b) S $\begin{cases} m = r + s \\ n = 2r - 3s \end{cases}$ (c) T $\begin{cases} \lambda = \alpha + 4\beta \\ \mu = 2\alpha + \beta \end{cases}$

   (d) U $\begin{cases} u = x \\ v = y \end{cases}$ (e) V $\begin{cases} R = 3r \\ S = 5s \end{cases}$ (f) W $\begin{cases} \lambda = \alpha + \beta \\ \mu = 2\beta \end{cases}$

4. Write transformations corresponding to the following matrices.

   (a) $\begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 5 \\ -2 & 3 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

5. For the transformations of question (3), find the following.

   (a) $R\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (b) $S\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (c) $T\begin{bmatrix} 4 \\ 4 \end{bmatrix}$

   (d) $U\begin{bmatrix} -2 \\ -3 \end{bmatrix}$ (e) $V\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ (f) $W\begin{bmatrix} 0 \\ -3 \end{bmatrix}$

6. For the transformations of question (3), find (x, y), (r, s), etc., such that

   (a) $R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, (b) $S\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, (c) $T\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$,

   (d) $U\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, (e) $V\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (f) $W\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

7. If T is the transformation corresponding to the matrix

$$\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

   find

   (a) $T\begin{bmatrix} 4 \\ -5 \end{bmatrix}$, (b) $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

8. If T is the transformation

$$T \quad \begin{cases} u = 3x - 2y \\ v = 4x + y \end{cases},$$

   show that every point in the uv-plane is the image of a unique point in the xy-plane; that is, show that a unique ordered pair (x, y) exists such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ for every pair (a, b) of real numbers.

9. Repeat question (8) for the transformation T of question (7).

10. Explain what is meant by saying that the transformation

$$T \quad \begin{cases} u = 3x + 5y \\ v = 2x + y \end{cases}$$

is a one-to-one mapping of the xy-plane onto the uv-plane. Show that it is.

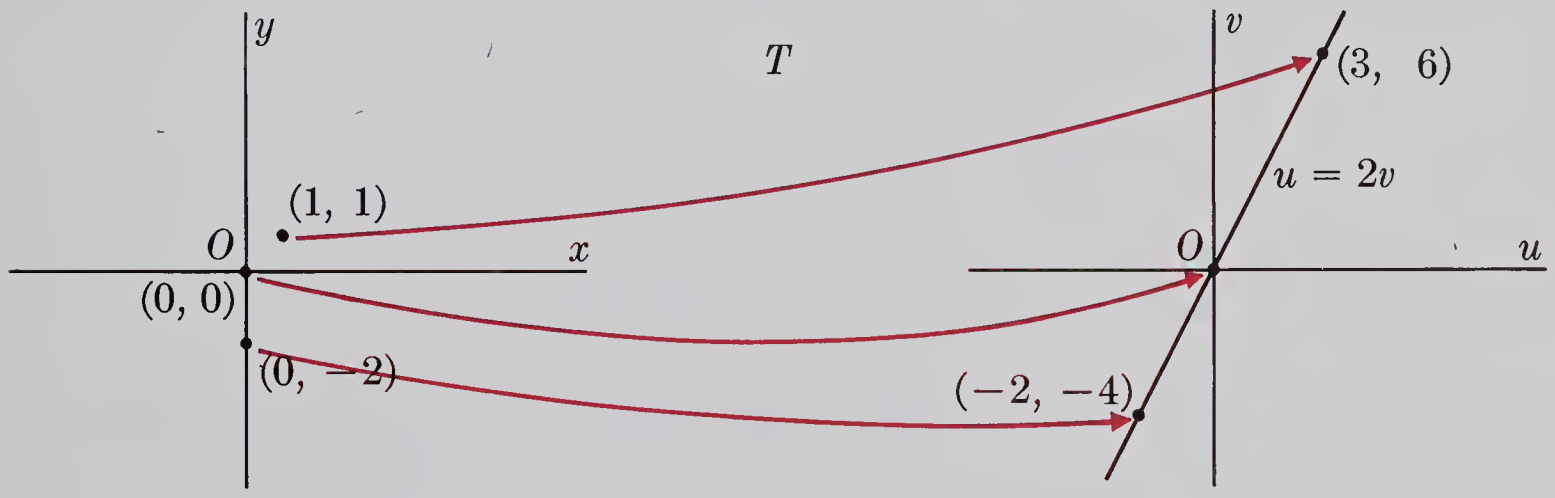
7.5 Linear Transformations of the Plane into the Plane

The linear transformations of the preceding section were all transformations of the xy-plane *onto* the uv-plane; that is, every point of the xy-plane had an image, and every point of the uv-plane was the image of some point under these transformations. Further, the transformations were all one-to-one transformations of the plane onto the plane; that is, each point in the xy-plane had a unique image, and each point in the uv-plane was the image of exactly one point under these transformations.

Now consider the transformation

$$T \quad \begin{cases} u = 2x + y \\ v = 4x + 2y \end{cases}.$$

It is evident that every point in the xy-plane will have an image under this transformation so that T is a transformation *of* the xy-plane. It is also evident that the images will be points with co-ordinates (u, v) such that $v = 2u$. This means that all the image points will be on the line with equation $v = 2u$ in the uv-plane.



Under this transformation, it is obvious that not all points of the uv-plane will be image points. Indeed, none of the points not on the line with equation $v = 2u$ will be image points. This transformation does not provide a one-to-one correspondence between the points of the xy-plane and the points of the uv-plane. This is not a transformation of the plane onto the plane, but rather a transformation of the plane *into* the plane.

The matrix associated with this transformation is

$$T = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}.$$

Example 1. Show that

$$S \begin{cases} u = -2x + 3y \\ v = 6x - 9y \end{cases}$$

is a transformation of the plane *into* the plane.

Solution:

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{with } v = -3u\,.$$

Thus all image points lie on the line in the uv-plane with equation $v = -3u$; points not on this line will not be image points under this transformation. This transformation is a transformation of the plane into the plane. The matrix associated with this transformation is

$$S = \begin{bmatrix} -2 & 3 \\ 6 & -9 \end{bmatrix}.$$

Example 2.

(a) For the transformation S in Example 1, find $S\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(b) Do any other points of the xy-plane have the image point found in (a)?

Solution:

(a)

$$S\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}.$$

(b) We readily check that

$$S\begin{bmatrix} 4 \\ 0 \end{bmatrix} = S\begin{bmatrix} 7 \\ 2 \end{bmatrix} = S\begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}.$$

As a matter of fact, the question asked in part (b) of Example 2 is equivalent to asking "Is the solution of the system of equations

$$\begin{aligned} -2x + 3y &= -8 \\ 6x - 9y &= 24 \end{aligned}$$

unique; that is, is the system independent or dependent?" We know that the system does have a solution, namely $(1, -2)$, so that the system is consistent. We may replace the system by an equivalent reduced system of the form

$$\begin{aligned} -2x + 3y &= -8\,, \\ 0y &= 0\,. \end{aligned}$$

Now we see that the system is dependent and has the solution set

$$\left\{ \left(\frac{3k + 8}{2}\,, k \right) \middle| \, k \in Re \right\}.$$

This solution set contains the particular solutions $(4, 0)$, $(7, 2)$, and $(10, 4)$ indicated above.

We also see that every point with co-ordinates $\left(\frac{3k+8}{2}, k\right)$ is mapped onto the point $(-8, 24)$ by S. But these points are all points with co-ordinates (x, y) such that

$$x = \frac{3y+8}{2},$$

that is, all points on the line $-2x + 3y = -8$. Thus this entire line is mapped onto the point $(-8, 24)$ by the transformation S.

Example 3. Find the set of points mapped onto the point $(1, -3)$ by the transformation S of Example 1.

Solution: The required set will consist of all points with co-ordinates (x, y) such that

$$-2x + 3y = 1 \quad \text{and} \quad 6x - 9y = -3\,;$$

that is, all points on the line with equation $-2x + 3y = 1$. The solution set of the equation $-2x + 3y = 1$ is

$$\left\{\left(\frac{3k-1}{2}, k\right) \middle| k \in Re\right\}.$$

We check our original statement by noting that

$$S\begin{bmatrix} \frac{3k-1}{2} \\ k \end{bmatrix} = \begin{bmatrix} -2\left(\frac{3k-1}{2}\right) + 3k \\ 6\left(\frac{3k-1}{2}\right) - 9k \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Example 4. Find the set of points mapped onto the point $(0, 0)$ by the transformation S of Example 1.

Solution: The set of points on the line whose equation is

$$-2x + 3y = 0$$

is the required set, since these points have co-ordinates (x, y) such that

$$u = -2x + 3y = 0$$

and

$$v = 6x - 9y = 0\,,$$

so that

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for all such points } (x, y)\,.$$

Finally, we note that under transformation S of Example 1,

line $-2x + 3y = -8$ is mapped onto point $(-8, 24)$;
line $-2x + 3y = 1$ is mapped onto point $(1, -3)$;
line $-2x + 3y = 0$ is mapped onto point $(0, 0)$.

In general, we see that the line whose equation is $-2x + 3y = k$ is mapped onto the point with co-ordinates $(k, -3k)$. Thus S is a many-to-one mapping of points of the xy-plane onto certain points of the uv-plane. Indeed, S sets up a one-to-one correspondence between the members of the family of parallel lines whose equations are $-2x + 3y = k$, $k \in Re$, and the points $(k, -3k)$ on the line $y = -3x$.

EXERCISE 7.5

Show that the following transformations are mappings of the xy-plane *into* the uv-plane.

1. $R \begin{cases} u = x - 2y \\ v = 3x - 6y \end{cases}$
2. $S \begin{cases} u = \frac{1}{2}x + y \\ v = x + 2y \end{cases}$
3. $T \begin{cases} u = -x + 4y \\ v = 2x - 8y \end{cases}$
4. $U \begin{cases} u = 3x - 2y \\ v = -x + \frac{2}{3}y \end{cases}$
5. $V \begin{cases} u = 3x - y \\ v = 6x - 2y \end{cases}$
6. $W \begin{cases} u = x + y \\ v = 4x + 4y \end{cases}$
7. $X \begin{cases} u = -x + \frac{1}{3}y \\ v = x - \frac{1}{3}y \end{cases}$
8. $Y \begin{cases} u = x + 2y \\ v = 0 \end{cases}$
9. $Z \begin{cases} u = 0 \\ v = 4x - y \end{cases}$
10. Find the set of points whose image under R is $R \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
11. Find the set of points whose image under T is $T \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.
12. Find the set of points whose image under V is $V \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.
13. Find the set of points whose image under X is $X \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.
14. Find the set of points whose image under Z is $Z \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
15. The transformations in questions (1) to (9) set up one-to-one correspondences between the members of certain families of parallel lines and the points of a line. Find the family of parallel lines and the corresponding points for each of these transformations.
16. Under what algebraic condition on the coefficients will the transformation

$$T \begin{cases} u = ax + by \\ v = cx + dy \end{cases}$$

be a transformation of the xy-plane *into* the uv-plane?

7.6. Transformations with Inverses

If a transformation T sets up a one-to-one correspondence between the points of the xy- and the uv-planes, that is, if T is a one-to-one transformation of the xy-plane onto the uv-plane, then there should be a corresponding one-to-one mapping T^* of the uv-plane onto the xy-plane. This mapping T^* should undo what mapping T does; that is, mapping T^* should be the inverse of mapping T.

Example 1. If T is the transformation

$$T \quad \begin{cases} u = 3x - 2y \\ v = 4x + y \end{cases},$$

find a transformation T^* by solving for x and y in terms of u and v. Show that T^* is the inverse of mapping T in the sense that if T maps point P onto point P^*, then T^* maps point P^* onto point P.

Solution: The result of question (8), Exercise 7.4 shows that T is a one-to-one transformation of the xy-plane onto the uv-plane. We solve the system

$$\begin{aligned} 3x - 2y &= u \\ 4x + y &= v \end{aligned}$$

in one of the usual ways (see Chapter 6) to obtain

$$T^* \quad \begin{cases} x = \frac{1}{11}u + \frac{2}{11}v \\ y = \frac{-4}{11}u + \frac{3}{11}v \end{cases}.$$

T^* is obviously a transformation of the uv-plane. Also, the image of point P with co-ordinates (a, b) under T is the point P^* with co-ordinates $(3a - 2b, 4a + b)$ since

$$T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a - 2b \\ 4a + b \end{bmatrix}.$$

We now find the image of point P^* under T^*; that is, we find

$$T^* \begin{bmatrix} 3a - 2b \\ 4a + b \end{bmatrix}.$$

But

$$\tfrac{1}{11}(3a - 2b) + \tfrac{2}{11}(4a + b) = a$$

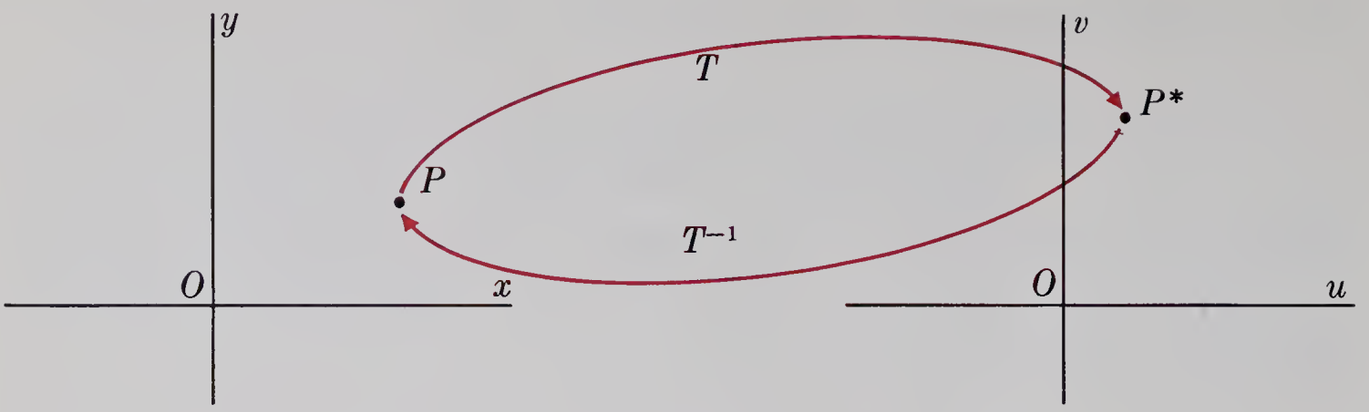
and

$$\tfrac{-4}{11}(3a - 2b) + \tfrac{3}{11}(4a + b) = b,$$

so that

$$T^* \begin{bmatrix} 3a - 2b \\ 4a + b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

This proves that if T maps point P onto point P^*, then T^* maps P^* back onto P. We call T^* the inverse transformation of T; we normally use the symbol T^{-1} to represent the inverse transformation of T.



As specific examples, we note that

$$T\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}1\\16\end{bmatrix} \quad \text{and} \quad T^{-1}\begin{bmatrix}1\\16\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix};$$

$$T\begin{bmatrix}5\\-2\end{bmatrix}=\begin{bmatrix}19\\18\end{bmatrix} \quad \text{and} \quad T^{-1}\begin{bmatrix}19\\18\end{bmatrix}=\begin{bmatrix}5\\-2\end{bmatrix};$$

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix} \quad \text{and} \quad T^{-1}\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

The transformations T and T^{-1} of this example both set up one-to-one correspondences between the sets of points of the xy- and uv-planes; T is a one-to-one transformation of the xy-plane onto the uv-plane and T^{-1} is a one-to-one transformation of the uv-plane onto the xy-plane.

Example 2. Find, if possible, the inverse transformation of the transformation

$$S \quad \begin{cases} u = x + 3y \\ v = 2x + 6y \end{cases}.$$

Solution: The transformation S does not produce a one-to-one correspondence between the points of the xy-plane and the points of the uv-plane. All points of the xy-plane have images which lie on the line $v = 2u$ in the uv-plane.

Indeed, S sets up a one-to-one correspondence between the members of the family of parallel lines $x + 3y = k$, $k \in Re$, and the points $(k, 2k)$ of the uv-plane. It seems reasonable then that, since the points of the line $x + 3y = k$ map onto the single point with co-ordinates $(k, 2k)$, there would be no way of knowing which point of the line to take as the image of the point with co-ordinates $(k, 2k)$ under an inverse transformation.

In addition, points not on the line $v = 2u$ would not have images under such an inverse transformation. Therefore it is clear that there can be no inverse for the transformation S.

The transformation

$$T \quad \begin{cases} u = ax + by \\ v = cx + dy \end{cases}$$

will have an inverse T^{-1} if and only if T is a one-to-one mapping of the xy-plane onto the uv-plane. We see the algebraic conditions on the coefficients a, b, c, d for T to have an inverse in the following theorem.

Theorem. The transformation

$$T \quad \begin{cases} u = ax + by \\ v = cx + dy \end{cases}$$

has an inverse if and only if $ad - bc \neq 0$.

Proof: The transformation T has an inverse T^{-1} if and only if we can solve

$$u = ax + by, \qquad v = cx + dy$$

for x and y in terms of u and v. We find that

$$x = \frac{du - bv}{ad - bc}, \qquad y = \frac{-cu + av}{ad - bc}$$

and so this yields the desired transformation T^{-1} if and only if

$$ad - bc \neq 0 .$$

We conclude that T is a one-to-one mapping of the xy-plane onto the uv-plane if and only if $ad - bc \neq 0$.

Example 3. Do the transformations

(a) $T \quad \begin{cases} u = 5x - 2y \\ v = -x + 3y \end{cases}$ (b) $S \quad \begin{cases} u = 3x - y \\ v = -9x + 3y \end{cases}$

have inverses?

Solution:

(a) Here $ad - bc$ has the value

$$(5)(3) - (-2)(-1) = 13 \neq 0 .$$

Therefore T has an inverse; T is a one-to-one mapping of the xy-plane onto the uv-plane.

(b) Here $ad - bc$ has the value

$$(3)(3) - (-1)(-9) = 0 .$$

Therefore S does not have an inverse; S is a mapping of the xy-plane into the uv-plane.

EXERCISE 7.6

State which of the following transformations do not have inverses and find the inverses of the remaining transformations.

1. $A \begin{cases} u = x + 3y \\ v = 3x + y \end{cases}$
2. $B \begin{cases} u = 3x + 2y \\ v = x - y \end{cases}$
3. $C \begin{cases} u = 3x - y \\ v = x - 2y \end{cases}$
4. $D \begin{cases} u = 3x \\ v = 2y \end{cases}$
5. $E \begin{cases} u = 3y \\ v = 2x \end{cases}$
6. $F \begin{cases} u = x - y \\ v = -2x + 2y \end{cases}$
7. $R \begin{cases} u = x + 2y \\ v = 0 \end{cases}$
8. $S \begin{cases} u = x + y \\ v = 3x \end{cases}$
9. $T \begin{cases} u = 2y \\ v = 2x - 3y \end{cases}$
10. $U \begin{cases} u = 0 \\ v = 3x - y \end{cases}$
11. $V \begin{cases} u = 3x + y \\ v = 2x - y \end{cases}$
12. $W \begin{cases} u = -x + 3y \\ v = 4x - 12y \end{cases}$
13. For the transformations in questions (1) to (12) that have inverses, find the images of the points with the co-ordinates
    (a) $(1, 2)$ (b) $(-3, 0)$ (c) $(2, -5)$ (d) $(2, -4)$
    and show in each case that the inverse transformation maps the image onto the original point.
14. The transformations we have been studying are called linear transformations. The points $P_1(2, 3)$, $P_2(-4, 7)$, and $P_3(-1, 5)$ are collinear. Show that the images of P_1, P_2, and P_3 are collinear under the transformations of questions (1) to (6).
15. Show that if points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ are collinear, and T is the transformation
$$T \begin{cases} u = ax + by \\ v = cx + dy \end{cases},$$
then $T(P_1)$, $T(P_2)$, and $T(P_3)$ are collinear points. What is the situation if $ad - bc = 0$?

7.7. Products of Transformations

The transformation
$$T \begin{cases} u = x - 2y \\ v = 2x + y \end{cases}$$
provides a one-to-one mapping of the xy-plane onto the uv-plane. ($ad - bc$ has the value 5.) Similarly, the transformation
$$S \begin{cases} r = 3u + v \\ s = u + 2v \end{cases}$$

provides a one-to-one mapping of the uv-plane onto the rs-plane. ($ad - bc$ has the value 5.) The transformation T maps every ordered pair (x, y) onto an ordered pair (u, v).

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

S maps every ordered pair (u, v) onto an ordered pair (r, s).

$$S\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}.$$

Then the performance of transformation T followed by the performance of transformation S has the total effect of mapping every ordered pair (x, y) onto an ordered pair (r, s), by using an intermediate step.

$$(x, y) \xrightarrow{\quad T \quad} (u, v) \xrightarrow{\quad S \quad} (r, s)$$

This combination of the two transformations T and S, namely transformation T followed by transformation S, is itself a transformation of the xy-plane into or onto the rs-plane. We call such a combination of two transformations T and S the product ST of the two transformations.

The product ST of two transformations T and S is the transformation equivalent to that obtained by carrying out transformation T and following this with transformation S (note the apparent reversal of the normal order). Then

$$ST\begin{bmatrix} x \\ y \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}.$$

Example 1. Find

(a) $ST\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $ST\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (c) $ST\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

for the transformations S and T of this section.

Solution:

(a) $$ST\begin{bmatrix} 1 \\ 1 \end{bmatrix} = S\left(T\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = S\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

(b) $$ST\begin{bmatrix} -2 \\ 5 \end{bmatrix} = S\left(T\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right) = S\begin{bmatrix} -12 \\ 1 \end{bmatrix} = \begin{bmatrix} -35 \\ -10 \end{bmatrix}.$$

(c) $$ST\begin{bmatrix} 3 \\ -4 \end{bmatrix} = S\left(T\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right) = S\begin{bmatrix} 11 \\ 2 \end{bmatrix} = \begin{bmatrix} 35 \\ 15 \end{bmatrix}.$$

Example 2. Find $ST \begin{bmatrix} x \\ y \end{bmatrix}$ for the transformations S and T of this section.

Solution:

$$(x, y) \xrightarrow{T} (u, v) = (x - 2y, 2x + y) .$$

$$\begin{aligned} (u, v) \xrightarrow{S} (r, s) &= (3u + v, u + 2v) \\ &= [3(x - 2y) + (2x + y), x - 2y + 2(2x + y)] \\ &= (5x - 5y, 5x) . \end{aligned}$$

Therefore,

$$ST \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x - 5y \\ 5x \end{bmatrix}.$$

The transformation ST is

$$ST \quad \begin{cases} r = 5x - 5y \\ s = 5x \end{cases}.$$

This provides a one-to-one mapping of the xy-plane onto the rs-plane since $ad - bc$ has the value 25. We may now check the results in Example 1.

(a)

$$ST \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 - 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

(b)

$$ST \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -10 - 25 \\ -10 \end{bmatrix} = \begin{bmatrix} -35 \\ -10 \end{bmatrix}.$$

(c)

$$ST \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 15 + 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 35 \\ 15 \end{bmatrix}.$$

Example 3. Find the product transformation AB for the transformations

$$B \quad \begin{cases} u = x - 3y \\ v = 2x + y \end{cases}, \qquad A \quad \begin{cases} r = -u + 2v \\ s = 2u - 4v \end{cases}.$$

Solution: We express r and s in terms of x and y as follows.

$$r = -u + 2v = -(x - 3y) + 2(2x + y) = 3x + 5y ,$$

$$s = 2u - 4v = 2(x - 3y) - 4(2x + y) = -6x - 10y .$$

Therefore,

$$AB \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 5y \\ -6x - 10y \end{bmatrix}.$$

The transformation AB is

$$AB \quad \begin{cases} r = 3x + 5y \\ s = -6x - 10y \end{cases}.$$

EXERCISE 7.7

1. Find $S\begin{bmatrix}-1\\1\end{bmatrix}$ and $T\left(S\begin{bmatrix}-1\\1\end{bmatrix}\right)$ for the following transformations.

   (a) $S\begin{cases}u=5x-y\\v=x+2y\end{cases}$ $T\begin{cases}r=3u-v\\s=u+v\end{cases}$

   (b) $S\begin{cases}u=2x+3y\\v=x-y\end{cases}$ $T\begin{cases}r=3u+2v\\s=-u+2v\end{cases}$

   (c) $S\begin{cases}u=-x+2y\\v=2x-4y\end{cases}$ $T\begin{cases}r=-u+4v\\s=3u-v\end{cases}$

   (d) $S\begin{cases}u=3x+y\\v=x-5y\end{cases}$ $T\begin{cases}r=2u-v\\s=-6u+3v\end{cases}$

2. Find $S\begin{bmatrix}2\\-3\end{bmatrix}$ and $T\left(S\begin{bmatrix}2\\-3\end{bmatrix}\right)$ for the transformations in question (1).

3. Find the products TS of the transformations in question (1). Check the results of questions (1) and (2) by finding $TS\begin{bmatrix}-1\\1\end{bmatrix}$ and $TS\begin{bmatrix}2\\-3\end{bmatrix}$ in each case.

4. Find the product transformation BA if A and B are the transformations

$$A\begin{cases}u=3x-2y\\v=x+3y\end{cases} \quad \text{and} \quad B\begin{cases}r=3u+v\\s=2u-v\end{cases}$$

5. Find the inverse transformations A^{-1} and B^{-1} of the transformations in question (4).

6. Find the inverse $(BA)^{-1}$ of the transformation in question (4).

7. Find the product transformation $A^{-1}B^{-1}$ of the transformations in question (4). Compare the result with that in question (6).

8. Repeat questions (4) to (7) for the transformations

$$A\begin{cases}u=5x+y\\v=2x\end{cases} \qquad B\begin{cases}r=3v\\s=u+v\end{cases}.$$

Chapter Summary

Definition of 2×2 and $m\times n$ real matrices · Dimensions of a matrix · Entries or components of a matrix · Row and column vectors · Equality of matrices · Addition of matrices · Multiplication of a matrix by a real number · Properties of addition of matrices: commutativity, associativity, existence of identity, existence of inverses · Linear transformations of the plane · Associated matrices · One-to-one transformations of the plane onto the plane · Linear transformations of the plane into the plane · Transformations with inverses · Products of transformations · ST is the transformation equivalent to the result of performing transformation T first, followed by transformation S

REVIEW EXERCISE 7

1. State (i) a_{12}, (ii) b_{21}, (iii) b_{22}, and find $a_{21}b_{11} + a_{22}b_{21}$ if
$$A = \left[a_{ij}\right] = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad B = \left[b_{ij}\right] = \begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix}.$$

2. For the matrices A and B of question (1), find
(a) $A + B$, (b) $(-1)B$, (c) $3A + 2B$.

3. Show that $A + B = B + A$ and $A + (B + C) = (A + B) + C$ for the matrices A and B of question (1) and
$$C = \begin{bmatrix} -5 & 4 \\ 1 & -7 \end{bmatrix}.$$

4. Find $A - B - C$ and $A - (B + C)$ for the matrices of question (3).

5. (a) Write a transformation T corresponding to the matrix
$$\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}.$$
(b) Find $T\begin{bmatrix} -5 \\ 1 \end{bmatrix}$.
(c) Find a and b such that
$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$
(d) Show that (x, y) exists such that
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$
for every ordered pair (m, n) of real numbers.

6. (a) Show that S is a transformation of the plane *into* the plane if
$$S \begin{cases} u = -x + 4y \\ v = 3x - 12y \end{cases}.$$
(b) Show that S sets up a one-to-one correspondence between the members of a certain family of parallel lines and the points of a line.

7. Find the inverses of the following transformations (if possible).
(a) $u = 3x + 2y$, $v = 3x - 2y$ (b) $u = 3x + y$, $v = 6x + y$ (c) $u = 3x + y$, $v = 6x + 2y$

8. For the following transformations find
(a) SR, (b) $T(SR)$, (c) TS, (d) $(TS)R$.
$$R \begin{cases} u = 2x + y \\ v = 3x - y \end{cases} \quad S \begin{cases} r = 3u + 2v \\ s = 6u + v \end{cases} \quad T \begin{cases} w = 2r \\ z = 3s \end{cases}$$

Chapter 8

MATRIX MULTIPLICATION

8.1 Definition of Matrix Multiplication

In this chapter, we shall reconsider "multiplication" of linear transformations and use it to define a corresponding multiplication of matrices. We shall then go on to study the properties and some of the applications of matrix multiplication.

We should point out that, although there are other possible ways of defining the product of two matrices, no use has been found for such definitions. On the other hand, the definition we shall give proves extremely useful.

In this section, we shall develop the rule for multiplying 2×2 real matrices. The method will apply in general whenever multiplication of matrices is possible.

We have seen that we have a one-to-one correspondence between 2×2 real matrices and transformations of the form

$$T \quad \begin{cases} u = ax + by \\ v = cx + dy \end{cases}.$$

Such a transformation corresponds to the matrix of coefficients

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the transformations

$$T \quad \begin{cases} u = x - 2y \\ v = 2x + y \end{cases} \qquad \text{and} \qquad S \quad \begin{cases} r = 3u + v \\ s = u + 2v \end{cases}$$

correspond to the matrices

$$T = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

respectively.

In Section 7.7, we defined the product transformation ST of two such transformations T and S, and, in Example 2 of that section, we found the product of these particular transformations T and S to be the transformation

$$ST \quad \begin{cases} r = 5x - 5y \\ s = 5x \end{cases}.$$

This transformation corresponds to the matrix

$$\begin{bmatrix} 5 & -5 \\ 5 & 0 \end{bmatrix},$$

and we define this matrix to be the product ST of the matrices corresponding to the transformations S and T. If

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix},$$

then

$$ST = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 \\ 5 & 0 \end{bmatrix}.$$

DEFINITION. The product ST of two matrices S and T is the matrix of coefficients of the product transformation ST of the two linear transformations corresponding to S and T.

Example 1. Find ST if

$$S = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}.$$

Solution: Let $T = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ correspond to the transformation

$$T \quad \begin{cases} u = 4x + y \\ v = -x + 2y \end{cases}$$

and $S = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}$ correspond to the transformation

$$S \quad \begin{cases} r = 5u - v \\ s = 2u + 3v \end{cases}.$$

Then, by definition, ST corresponds to the product transformation.

$$r = 5(4x + y) - (-x + 2y) = 21x + 3y$$

and

$$s = 2(4x + y) + 3(-x + 2y) = 5x + 8y,$$

so that ST is the transformation

$$ST \quad \begin{cases} r = 21x + 3y \\ s = 5x + 8y \end{cases}.$$

Therefore, by our definition,

$$ST = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 3 \\ 5 & 8 \end{bmatrix}.$$

Example 2. Find TS for the matrices of Example 1.

Solution: Let $S = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}$ correspond to the transformation

$$S \quad \begin{cases} u = 5x - y \\ v = 2x + 3y \end{cases}$$

and $T = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ correspond to the transformation

$$T \quad \begin{cases} r = 4u + v \\ s = -u + 2v \end{cases}.$$

Then TS corresponds to the product transfomation.

$$r = 4(5x - y) + (2x + 3y) = 22x - y$$

and

$$s = -(5x - y) + 2(2x + 3y) = -x + 7y\,,$$

so that TS is the transformation

$$TS \quad \begin{cases} r = 22x - y \\ s = -x + 7y \end{cases}.$$

Therefore, by definition,

$$TS = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -1 \\ -1 & 7 \end{bmatrix}.$$

We see from Examples 1 and 2 that matrices ST and TS are not equal, and we conclude that, in general, multiplication of matrices, as we have defined it, is not commutative. This does not mean that AB is never equal to BA; it is true that $AB = BA$ for certain matrices, but not for all matrices A and B.

EXERCISE 8.1

Find AB and BA for the following pairs of matrices.

1. $A = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 3 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{-4}{11} & \frac{3}{11} \end{bmatrix}$

6. $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

8.2. Rule for Multiplication of Matrices

We find it convenient to develop a rule for multiplying 2×2 matrices rather than setting up the corresponding transformations each time as in the previous section. We consider two general matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

and find their product, using the definition given in Section 7.8 for the product of the corresponding transformations.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

correspond to the transformation

$$\begin{cases} u = a_{11}x + a_{12}y \\ v = a_{21}x + a_{22}y \end{cases},$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

correspond to the transformation

$$\begin{cases} r = b_{11}u + b_{12}v \,. \\ s = b_{21}u + b_{22}v \end{cases}$$

Then matrix BA corresponds to the product transformation.

$$\begin{aligned} r &= b_{11}(a_{11}x + a_{12}y) + b_{12}(a_{21}x + a_{22}y) \\ &= (b_{11}a_{11} + b_{12}a_{21})x + (b_{11}a_{12} + b_{12}a_{22})y \end{aligned}$$

and

$$\begin{aligned} s &= b_{21}(a_{11}x + a_{12}y) + b_{22}(a_{21}x + a_{22}y) \\ &= (b_{21}a_{11} + b_{22}a_{21})x + (b_{21}a_{12} + b_{22}a_{22})y \,, \end{aligned}$$

so that the product transformation is

$$\begin{cases} r = (b_{11}a_{11} + b_{12}a_{21})x + (b_{11}a_{12} + b_{12}a_{22})y \\ s = (b_{21}a_{11} + b_{22}a_{21})x + (b_{21}a_{12} + b_{22}a_{22})y \end{cases}.$$

Therefore, by definition,

$$\begin{aligned} BA &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}. \end{aligned}$$

This formula may appear quite formidable as one for the product of the matrices B and A until we point out a fairly simple rule for obtaining the terms of the product matrix.

If we let BA be the matrix C, with

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

we see that

(i) c_{11} is the sum of the products of corresponding terms of the *first* row of B and the *first* column of A;

$$[b_{11} \quad b_{12}] \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$

column 1

row 1

that is,

$$c_{11} = b_{11}a_{11} + b_{12}a_{21};$$

(ii) c_{21} is the sum of the products of corresponding terms of the *second* row of B and the *first* column of A;

$$[b_{21} \quad b_{22}] \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \textit{column 1}$$

row 2

that is,

$$c_{21} = b_{21}a_{11} + b_{22}a_{21};$$

(iii) c_{12} is the sum of the products of corresponding terms of the *first* row of B and the *second* column of A;

$$[b_{11} \quad b_{12}] \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \quad \textit{column 2}$$

row 1

that is,

$$c_{12} = b_{11}a_{12} + b_{12}a_{22};$$

(iv) c_{22} is the sum of the products of corresponding terms of the *second* row of B and the *second* column of A;

$$[b_{21} \quad b_{22}] \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \quad \textit{column 2}$$

row 2

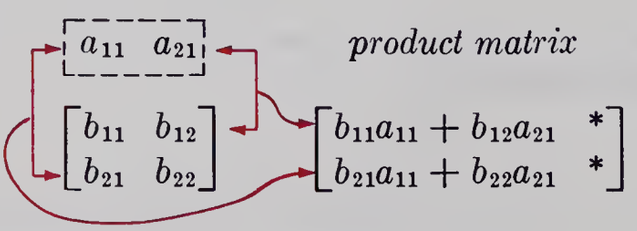
that is,

$$c_{22} = b_{21}a_{12} + b_{22}a_{22}\,.$$

The finding of the product matrix can be made even more mechanical by noting that the above results may be obtained for the product

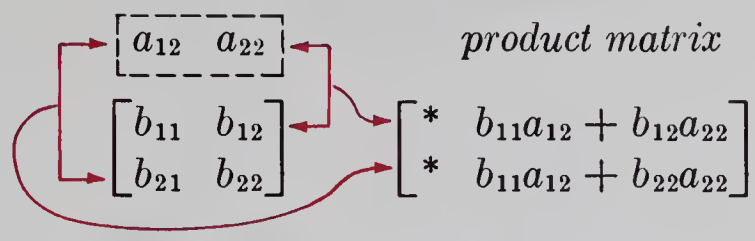
$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

by picturing the first column of the right matrix picked up and placed horizontally above the left matrix as shown below.



The sums of the products of corresponding elements for this row and the first and second rows of B then produce the terms of the first column of the product matrix as indicated.

Now picture the second column of the right matrix picked up and placed horizontally above the left matrix as shown below.

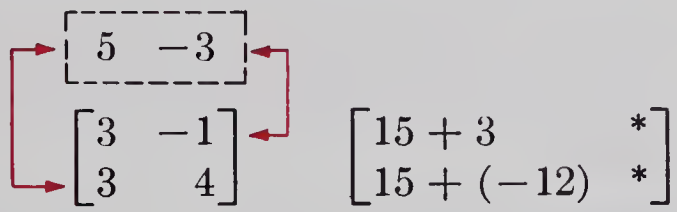


Again, the sums of the products of corresponding elements of this row and the first and second rows of B produce the terms of the second column of the product matrix as indicated.

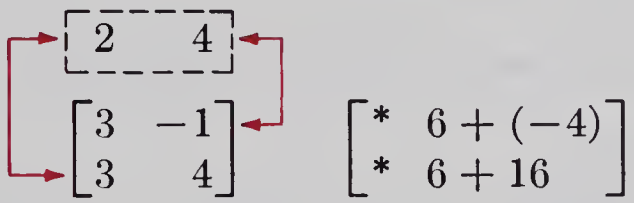
Example 1. Find AB if

$$A = \begin{bmatrix} 3 & -1 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}.$$

Solution: The first column of AB is obtained as indicated in the following scheme



and the second column of AB is obtained similarly.



Therefore,

$$\begin{bmatrix} 3 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 2 \\ 3 & 22 \end{bmatrix}.$$

Of course, in practice, the steps outlined in detail above are performed mentally.

Example 2. Find BA for the matrices A and B of Example 1.

Solution:

$$BA = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3(5) + 3(2) & (-1)5 + 4(2) \\ 3(-3) + 3(4) & (-1)(-3) + 4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 3 \\ 3 & 19 \end{bmatrix}.$$

Again we see that $AB \neq BA$ for the matrices A and B of Examples 1 and 2.

Example 3. If $A^2 = AA$, find A^2 if

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A^2 = AA &= \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3(3) + (-2)4 & 4(3) + 1(4) \\ 3(-2) + (-2)1 & 4(-2) + 1(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 16 \\ -8 & -7 \end{bmatrix}. \end{aligned}$$

Finally, we see that, in the product matrix

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

we have

c_{ij} = the sum of the products of corresponding terms of the ith row of the left matrix and the jth column of the right matrix.

$$\begin{aligned} &= b_{i1}a_{1j} + b_{i2}a_{2j} \\ &= \sum_{k=1}^{2} b_{ik}a_{kj}\,. \end{aligned}$$

Then, abbreviating A by $[a_{ij}]$ and B by $[b_{ij}]$, we may write

$$\begin{aligned} BA &= [b_{ij}][a_{ij}] \\ &= \left[\sum_{k=1}^{2} b_{ik}a_{kj}\right]. \end{aligned}$$

This definition may be extended to the product of matrices other than 2×2 matrices under certain conditions of compatibility of the two matrices.

Example 4. Assuming that it is possible to extend our rule of multiplication (taking sums of the products of elements of rows of the *left-hand matrix* and the corresponding elements of the columns of the *right-hand matrix*), find, if possible, AB and BA if

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & -3 \\ 2 & 3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Solution:

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & -3 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) + 1(-2) + 2(4) \\ (-1)(3) + 4(-2) + (-3)(4) \\ 2(3) + 3(-2) + 0(4) \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ -23 \\ 0 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & -3 \\ 2 & 3 & 0 \end{bmatrix}.$$

Here, the rows of the left matrix contain only one element, while the columns of the right matrix contain three elements. It is impossible to multiply corresponding elements and so it is impossible to find BA.

EXERCISE 8.2

1. If

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix},$$

find

(a) AB, (b) BA, (c) A^2, (d) B^2, (e) $A^2 + A^2B$.

2. If

$$P = \begin{bmatrix} -1 & 5 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix},$$

find

(a) PQ, (b) QP, (c) P^2, (d) P^3, (e) P^4.

3. If

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix},$$

find

(a) AB, (b) BC, (c) CA, (d) $A^2 + B^2$, (e) $A - BC$.

4. If

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

find

(a) AI, (b) IA, (c) I^2, (d) A^2, (e) A^2I.

5. If

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

find

(a) AB, (b) BA, (c) B^2, (d) A^2, (e) BA^2.

6. If
$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix},$$
find AB and BA, and comment on the result.

7. Find the product AB, where possible, for the following A and B.

(a) $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

(b) $A = [1 \quad 5], \quad B = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ -1 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 6 & 4 \\ -1 & 2 \end{bmatrix}, \quad B = [3 \quad 1]$

(e) $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

(f) $A = [1 \quad 2 \quad -1], \quad B = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

(g) $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -3 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}$

(h) $A = [4 \quad 2 \quad -4], \quad B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 5 & 0 \end{bmatrix}$

(i) $A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

(j) $A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

8. If
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
find A^2, B^2, and C^2. How do your results differ from those obtained in similar situations for the real numbers?

9. Is it possible to extend our rule of multiplication (multiplying elements of rows of the left matrix by elements of columns of the right matrix) to find the products AB and BA of two matrices if

(a) A is a 2×2 matrix, B is a 2×1 matrix?

(b) A is a 1×2 matrix, B is a 2×2 matrix?

(c) B and A are both 3×3 matrices?

(d) B is a 3×2 matrix, A is a 2×3 matrix?

(e) B and A are both 3×2 matrices?

10. If B is an $m \times n$ matrix and A is an $r \times s$ matrix, under what condition will it be possible to form the product BA? the product AB?

8.3. Algebraic Properties of Multiplication of Matrices

In this section, we shall examine the algebraic properties of matrix multiplication as applied to the set of all 2×2 real matrices.

(1) *Closure* It is always possible to find the product of two 2×2 real matrices, although, as we have seen, it is not always possible to find the product of any two matrices. By our definition, the product of two 2×2 real matrices is again a 2×2 real matrix. Thus the set of all 2×2 real matrices is closed under matrix multiplication.

(2) *Commutativity* We have seen from a number of examples that multiplication of 2×2 real matrices is not commutative in general. It may be true that $AB = BA$ for some special matrices A and B, but, in general, $AB \neq BA$.

(3) *Associativity* Multiplication of 2×2 real matrices is associative; for all 2×2 real matrices A, B, and C,

$$A\,(BC) = (AB)C\,.$$

We shall not prove this statement but we shall check it in Example 1 and in the exercises.

Example 1. Show that $A\,(BC) = (AB)C$ if

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}.$$

Solution:

$$BC = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 17 & -4 \end{bmatrix}.$$

$$A\,(BC) = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 17 & -4 \end{bmatrix} = \begin{bmatrix} -47 & 14 \\ 76 & -12 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -8 & 7 \\ 14 & -6 \end{bmatrix}.$$

$$(AB)C = \begin{bmatrix} -8 & 7 \\ 14 & -6 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -47 & 14 \\ 76 & -12 \end{bmatrix}.$$

Therefore,

$$(AB)C = A(BC)$$

for these matrices A, B, and C.

(4) *Multiplicative Identity* The matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the multiplicative identity for the set of 2×2 real matrices. Certainly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so that

$$AI = IA = A$$

for all 2×2 real matrices A.

(5) *Distributivity* Multiplication of matrices is distributive over addition in the set of all 2×2 real matrices. Again we shall not formally prove this statement but we shall check it in Example 2 and in the exercises.

Example 2. Show that $A(B + C) = AB + AC$ for the matrices of Example 1.

Solution:

$$B + C = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 1 \\ 2 & 0 \end{bmatrix}.$$

$$A(B + C) = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 20 & 2 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -8 & 7 \\ 14 & -6 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & -6 \\ 6 & 8 \end{bmatrix}.$$

$$AB + AC = \begin{bmatrix} -8 & 7 \\ 14 & -6 \end{bmatrix} + \begin{bmatrix} 8 & -6 \\ 6 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 20 & 2 \end{bmatrix}.$$

Therefore,

$$A(B + C) = AB + AC$$

for these matrices A, B, and C.

(6) *Divisors of Zero* We know that the product ab of two real numbers can equal zero if and only if at least one of a or b is zero. We say that the set of real numbers does not contain divisors of zero. This is not true for the set of 2×2 real matrices. We recall that the zero matrix in this set is the matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This matrix is the additive identity in this set. It is possible to have $AB = O$ and yet neither A nor B be the zero matrix. This is exemplified in Example 3. The set of 2×2 real matrices contains divisors of zero.

Example 3. Find AB if

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 6 \\ 2 & -4 \end{bmatrix}.$$

Solution:

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} -3 & 6 \\ 2 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= O.$$

Note that $A \neq O$ and $B \neq O$.

EXERCISE 8.3

Find
(a) $A(BC)$, (b) $(AB)C$, (c) $A(B+C)$, (d) $AB+AC$
for the following matrices A, B, and C.

1. $A = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$

5. If
$$A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix},$$
find
(a) $A(AB)$, (b) A^2B, (c) $A(A+B)$, (d) A^2+AB.

6. If
$$A = \begin{bmatrix} 5 & 0 \\ 2 & 0 \end{bmatrix},$$
find a matrix $B \neq O$ such that $AB = O$.

7. If
$$A = \begin{bmatrix} 6 & -3 \\ 0 & 0 \end{bmatrix},$$
find a matrix $B \neq O$ such that $AB = O$.

8. If
$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix},$$
find a matrix $B \neq O$ such that $AB = O$.

9. If
$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix},$$
where a, b, c, d, e, f are real numbers, prove that $A(BC) = (AB)C$.

8.4. The Inverse of a Matrix

The identity matrix
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

corresponds to the identity linear transformation

$$I \quad \begin{cases} u = x + 0y \\ v = 0x + y \end{cases},$$

the transformation that maps every point onto itself; that is,

$$I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now if T is a linear transformation such that

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix},$$

and if it is possible to find the inverse transformation T^{-1} such that

$$T^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

then the product $T^{-1}T$ is such that

$$T^{-1}T \begin{bmatrix} x \\ y \end{bmatrix} = T^{-1}\left(T \begin{bmatrix} x \\ y \end{bmatrix}\right) = T^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix};$$

that is,

$$T^{-1}T = I\,.$$

Similarly,

$$TT^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = T\left(T^{-1} \begin{bmatrix} u \\ v \end{bmatrix}\right) = T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix},$$

so that again TT^{-1} is the identity linear transformation in the form

$$\begin{cases} x = u + 0v \\ y = 0u + v \end{cases}.$$

(Recall that we have already pointed out that the letters or variables or placeholders used are just dummies, so that the transformations

$$\begin{cases} u = x + 0y \\ v = 0x + y \end{cases} \quad \text{and} \quad \begin{cases} x = u + 0v \\ y = 0u + v \end{cases}$$

are essentially the same.)

If matrix T corresponds to transformation T and matrix S to transformation T^{-1}, then TS corresponds to TT^{-1} and ST to $T^{-1}T$ so that

$$TS = ST = I\,,$$

since I corresponds to the identity transformation.

S is the multiplicative inverse of T, and we use the symbol T^{-1} for this matrix. (Of course, T is also the inverse of S; $T = S^{-1}$.) Thus, a matrix will have an inverse if the corresponding linear transformation has an inverse.

Example 1. Find the inverse of the matrix

$$T = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}.$$

Solution: Matrix T corresponds to the transformation

$$T \quad \begin{cases} u = 2x - 3y \\ v = x + 6y \end{cases}.$$

We may solve for x and y in terms of u and v and obtain the inverse transformation

$$T^{-1} \quad \begin{cases} x = \frac{2}{5}u + \frac{1}{5}v \\ y = -\frac{1}{15}u + \frac{2}{15}v \end{cases}.$$

According to our reasoning, the matrix

$$S = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{15} & \frac{2}{15} \end{bmatrix}$$

should be the inverse of T. We check by multiplication.

$$\begin{aligned} TS &= \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{15} & \frac{2}{15} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I. \end{aligned}$$

$$\begin{aligned} ST &= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{15} & \frac{2}{15} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I. \end{aligned}$$

Therefore, $S = T^{-1}$ by definition.

We may always find the inverse of a 2×2 real matrix by the method of Example 1, provided we can find the inverse of the corresponding transformation, and we have seen earlier that some transformations do not have inverses. In the proof of the following theorem, we will apply the technique of Example 1 in general to see under what condition a 2×2 real matrix will have an inverse, and to see how to write the inverse immediately if it exists.

Theorem. The matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has an inverse

$$T^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

if and only if $ad - bc \neq 0$.

Proof: Matrix T corresponds to the transformation

$$T \quad \begin{cases} u = ax + by \\ v = cx + dy \end{cases}.$$

Then

$$cu = acx + bcy\,,$$
$$av = acx + ady\,,$$

and

$$av - cu = (ad - bc)\,y\,.$$

Also,

$$du = adx + bdy\,,$$
$$bv = bcx + bdy\,,$$

and

$$du - bv = (ad - bc)\,x\,.$$

Then

$$T^{-1} \quad \begin{cases} x = \dfrac{d}{ad - bc}\,u + \dfrac{-b}{ad - bc}\,v \\ y = \dfrac{-c}{ad - bc}\,u + \dfrac{a}{ad - bc}\,v \end{cases}$$

is the inverse transformation of T if and only if $ad - bc \neq 0$, as we saw in Section 7.6. We conclude that

$$T^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

is the inverse of T if and only if $ad - bc \neq 0$.

Check:

$$TT^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{ad - bc}{ad - bc} & \frac{-ab + ab}{ad - bc} \\ \frac{cd - cd}{ad - bc} & \frac{-bc + ad}{ad - bc} \end{bmatrix} \qquad \text{(provided } ad - bc \neq 0\text{)}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I .$$

The check that $T^{-1}T = I$ is left to the reader. If $ad - bc = 0$, so that the matrix does not have an inverse, the matrix is said to be non-invertible; otherwise, the matrix is invertible.

Example 2. Does the matrix

$$A = \begin{bmatrix} 5 & -4 \\ 1 & -2 \end{bmatrix}$$

have an inverse? If so, find A^{-1}.

Solution: We note that for this matrix $ad - bc$ has the value $5(-2) - (-4)(1) = -6$; hence, A^{-1} exists. From our expression for A^{-1}, we see that

$$A^{-1} = \begin{bmatrix} \frac{-2}{-6} & \frac{4}{-6} \\ \frac{-1}{-6} & \frac{5}{-6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{6} & \frac{-5}{6} \end{bmatrix}.$$

Check:

$$AA^{-1} = \begin{bmatrix} 5 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{6} & \frac{-5}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I .$$

Thus, A^{-1} is the required inverse.

Example 3. Find the inverse of the matrix

$$B = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix}.$$

Solution: $ad - bc$ has the value $(-1)(-8) - (4)(2) = 0$. This matrix does not have an inverse. B is a non-invertible matrix.

The technique used in the proof of the theorem of this section, that is, in the attempt to find the inverse of a linear transformation, may be used in general to determine under what conditions a given $m \times n$ matrix has an inverse.

EXERCISE 8.4

In questions (1) to (4), find the inverse of the corresponding linear transformation (as in Example 1) in order to find the inverse of the given matrix.

1. $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} -3 & 1 \\ 5 & 0 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$ 4. $\begin{bmatrix} 2 & -5 \\ 1 & 2 \end{bmatrix}$

Determine which of the following matrices have inverses and find the inverse when it exists.

5. $\begin{bmatrix} -1 & 3 \\ 5 & -15 \end{bmatrix}$ 6. $\begin{bmatrix} -1 & 3 \\ -15 & 5 \end{bmatrix}$ 7. $\begin{bmatrix} 1 & -3 \\ -15 & 5 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & -3 \\ -5 & 15 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$ 10. $\begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$ 11. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ 12. $\begin{bmatrix} 0 & 4 \\ -3 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ 14. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 15. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ 16. $\begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{3} & 1 \end{bmatrix}$

17. If a and d are nonzero, find the inverse of $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

18. If b and c are nonzero, find the inverse of $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$.

19. If $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, find

(a) AB (b) $(AB)^{-1}$ (c) A^{-1} (d) B^{-1}

and check that $(AB)^{-1} = B^{-1}A^{-1}$ for these matrices.

20. If A, B, and AB all have inverses, prove that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

8.5. The Linear Matrix Equation $AX = B$

We have mentioned earlier that a matrix may contain any number of rows and columns, and that one-rowed matrices, such as

$$[3 \quad 5], \qquad [4 \quad 0 \quad -3], \qquad [5 \quad 0 \quad 0 \quad 4],$$

are also called row vectors, and one-columned matrices, such as,

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \qquad \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

are also called column vectors. Thus a row vector is a $1 \times n$ matrix and a column vector is an $m \times 1$ matrix.

In defining multiplication of matrices, we pointed out that the product AB of two matrices A and B could always be found if our technique of "applying columns of the right matrix to rows of the left matrix and multiplying corresponding elements" could be used. This definition implied certain restrictions on the dimensions of A and B.

Example 1. Find AB if

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Solution: By our technique for multiplying matrices,

$$AB = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4(5) + (-3)(-1) \\ 4(3) + (-3)(2) \end{bmatrix} = \begin{bmatrix} 23 \\ 6 \end{bmatrix}.$$

Example 2. Find BA in Example 1.

Solution:

$$BA = \begin{bmatrix} 4 \\ -3 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix},$$

If we apply our multiplication technique, we arrive at the following.

$$\begin{bmatrix} 5 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

The elements of a column of A do not pair off with the elements of a row of B; hence, the product BA does not exist.

Example 3. Find AX if

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution:

$$AX = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}.$$

Example 4. Under what conditions will

$$\begin{bmatrix} 5x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}?$$

Solution: We know that two matrices are equal if and only if they have the same dimensions and corresponding components are equal. These matrices are both 2×1 matrices and so are equal if and only if

$$5x_1 - x_2 = 4$$

and

$$3x_1 + 2x_2 = -7\,.$$

We see now from the results of Examples 3 and 4 that the system of equations

$$\begin{aligned} 5x_1 - x_2 &= 4 \\ 3x_1 + 2x_2 &= -7 \end{aligned}$$

is equivalent to the single linear matrix equation $AX = B$ with

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 4 \\ -7 \end{bmatrix};$$

that is,

$$\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$$

This result can be generalized; a system of linear equations can always be replaced by a single linear matrix equation.

Example 5. Replace the system of equations

$$\begin{aligned} 3x_1 - 7x_2 &= 5 \\ x_1 + 5x_2 &= -3 \end{aligned}$$

by a single linear matrix equation.

Solution: Choose matrix A as the matrix of coefficients

$$A = \begin{bmatrix} 3 & -7 \\ 1 & 5 \end{bmatrix}.$$

Then

$$AX = \begin{bmatrix} 3 & -7 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 7x_2 \\ x_1 + 5x_2 \end{bmatrix},$$

so that the given system may be written as $AX = B$ with

$$A = \begin{bmatrix} 3 & -7 \\ 1 & 5 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 5 \\ -3 \end{bmatrix};$$

that is,

$$\begin{bmatrix} 3 & -7 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

EXERCISE 8.5

1. Find the products of the following pairs of matrices.

(a) $\begin{bmatrix} -1 & 5 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 5 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}$ (h) $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$

(i) $\begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} s \\ t \end{bmatrix}$ (j) $\begin{bmatrix} 6 & -1 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} m \\ n \end{bmatrix}$

2. Write the following systems of equations as single linear matrix equations.

(a) $6x_1 - x_2 = 3$
$x_1 + 4x_2 = 5$

(b) $2x + 5y = 6$
$x - 4y = -3$

(c) $3a + 2b = 5$
$a - 6b = -2$

(d) $2m - 7n = 0$
$5m + n = -3$

(e) $4r - s = 0$
$r + 3s = 0$

(f) $5u - 2v = -1$
$2u + 3v = 5$

3. For what values of k will $AU = V$ if

(a) $A = \begin{bmatrix} -3 & k \\ 2 & 4 \end{bmatrix}, \quad U = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$?

(b) $A = \begin{bmatrix} 2 & 5 \\ -k & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$?

(c) $A = \begin{bmatrix} 4 & 3k \\ -k & 5 \end{bmatrix}, \quad U = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} -1 \\ 11 \end{bmatrix}$?

4. If

$$A = \begin{bmatrix} 4 & 7 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad W = [1 \quad 4],$$

find, if possible, the products WA and AW.

8.6. The Solution of $AX = B$

In this section, we consider the linear matrix equation $AX = B$. We show that this equation has a (unique) solution if the matrix A is invertible (has an inverse). Thus the linear matrix equation $AX = B$ compares with the real linear equation $ax = b$ $(a, b \in Re)$ which has a unique solution if a^{-1} exists.

Example 1. Express the system of equations

$$\begin{aligned} 3x_1 - x_2 &= 5 \\ 2x_1 + 6x_2 &= -3 \end{aligned}$$

as a linear matrix equation.

Solution: We may write this system as $AX = B$ with

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 5 \\ -3 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Example 2. Find A^{-1} for the matrix A of Example 1.

Solution: $ad - bc$ has the value $3(6) - (-1)(2) = 20$; therefore A^{-1} exists. We find that

$$A^{-1} = \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{20} \end{bmatrix}.$$

Example 3. Multiply both sides of the matrix equation in Example 1 *on the left* by A^{-1} from Example 2, and solve the resulting expression for x_1 and x_2.

Solution:

$$\begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{20} \end{bmatrix} \left(\begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Multiplication of matrices is associative, and so we may write

$$\left(\begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

The product matrix in parentheses is I (why?), and we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{27}{20} \\ \frac{-19}{20} \end{bmatrix};$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{27}{20} \\ \frac{-19}{20} \end{bmatrix},$$

and so $x_1 = \frac{27}{20}$ and $x_2 = \frac{-19}{20}$ (why?). Thus

$$\left\{ \left(\tfrac{27}{20}, \tfrac{-19}{20} \right) \right\}$$

is the solution set of the system of equations in Example 1.

The above examples show how a system of linear equations can be solved if the corresponding linear matrix equation $AX = B$ can be solved.

Theorem. The linear matrix equation $AX = B$ can be solved if A^{-1} exists.

Proof: Suppose $AX = B$ and A^{-1} exists. Then

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B\,. \\ (A^{-1}A)X &= A^{-1}B\,. \\ IX &= A^{-1}B\,. \\ X &= A^{-1}B\,. \end{aligned}$$

Thus $A^{-1}B$ is the solution of $AX = B$.

Check:

$$A\,(A^{-1}B) = (AA^{-1})B = IB = B\,.$$

Note that in the proof of the theorem, we were careful to multiply both members of the equation $AX = B$ *on the left* by A^{-1} to obtain $A^{-1}(AX) = A^{-1}B$. Remember that multiplication of matrices is not necessarily commutative so that if $A = B$, then $AC = BC$, but $AC \neq CB$ necessarily.

Example 4. Use the method of this section to solve the system of equations

$$\begin{aligned} 3x_1 + 5x_2 &= 4\,, \\ 2x_1 - x_2 &= 0\,. \end{aligned}$$

Solution: We write the system in the form $AX = B$.

$$\begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$

and find

$$A^{-1} = \begin{bmatrix} \frac{1}{13} & \frac{5}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix}.$$

Then

$$\begin{bmatrix} \frac{1}{13} & \frac{5}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{13} & \frac{5}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{13} \\ \frac{8}{13} \end{bmatrix},$$

so that $\{(\frac{4}{13}, \frac{8}{13})\}$ is the required solution.

Example 5. Solve the system of equations

$$\begin{aligned} 2x_1 - x_2 &= 5\,, \\ -4x_1 + 2x_2 &= -3\,. \end{aligned}$$

Solution: If we write this system in the form $AX = B$ with

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix},$$

we find that A^{-1} does not exist. ($ad - bc$ has value 0.) Hence, we cannot use the present method to solve this system. We find, using other reasoning, that this system does not have a solution.

EXERCISE 8.6

Express the following systems of equations as linear matrix equations of the form $AX = B$, and solve the system if A^{-1} exists.

1. $3x_1 - 5x_2 = 3$
   $2x_1 + x_2 = 5$

2. $4x - y = 6$
   $x - 2y = -1$

3. $5a - 2b = -3$
   $2a + b = 4$

4. $3x_1 - x_2 = 7$
   $-12x_1 + 4x_2 = -3$

5. $3x_1 - x_2 = 7$
   $-12x_1 - 4x_2 = -3$

6. $4a + b = 0$
   $2a - b = -6$

7. $m + 5n = 6$
   $3m - 2n = -4$

8. $5s - 6t = 4$
   $3s - t = 0$

9. $6w - z = 2$
   $3w + 2z = 5$

10. $4a + 3b = m$
    $5a - 2b = n$

11. Show that the solution of $AX = B$ is unique if A^{-1} exists.

8.7. An Alternate Method for Finding A^{-1}.

The system of equations

$$\begin{aligned} 3x - y &= u \\ 2x + 5y &= v \end{aligned} \tag{1}$$

can be written as the single linear matrix equation

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix},$$

and, indeed, in the form

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{2}$$

We shall illustrate in the following examples a general result which will enable us to establish an algorithm for finding the inverse of an invertible matrix.

Example 1. Interchange the equations of system (1) and write the resulting system as a single linear matrix equation as in (2).

Solution: If we interchange equations in (1), we obtain the following equivalent system.

$$\begin{aligned} 2x + 5y &= v \,. \\ 3x - y &= u \,. \end{aligned}$$

The reader is asked to check that this system can be written as the single linear matrix equation

$$\begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Example 2. Multiply the second equation of system (1) by k, and write the resulting system as a single linear matrix equation as in (2).

Solution: If we multiply the second equation of (1) by k, we obtain the following equivalent system.

$$\begin{aligned} 3x - y &= u \,. \\ 2kx + 5ky &= kv \,. \end{aligned}$$

The reader is asked to check that this system can be written as the single linear matrix equation

$$\begin{bmatrix} 3 & -1 \\ 2k & 5k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Example 3. Add k times the first equation to the second equation in (1), and write the resulting system as a single linear matrix equation.

Solution: The resulting system is

$$\begin{aligned} 3x - y &= u \\ (2 + 3k)x + (5 - k)y &= v + ku \end{aligned}.$$

The reader is asked to check that this system can be written as the single linear equation

$$\begin{bmatrix} 3 & -1 \\ 2 + 3k & 5 + (-1)k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The examples above illustrate the following result. Any elementary row operation on a system of equations

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned} \tag{3}$$

is performed simultaneously on the matrices of coefficients in the corresponding linear matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{4}$$

Now perform, if possible, a sequence of elementary row operations on (3) so that (4) assumes the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix};$$

this corresponds to the system

$$\begin{aligned} x &= eu + fv \\ y &= gu + hv \end{aligned},$$

and so we will have inverted the system (3) and thus will have found the inverse of the original matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

This gives us the following rule.

Rule for Finding the Inverse of a Matrix

Let A be the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Form the matrix

$$B = \begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

by adjoining the matrices of coefficients in (4). Perform a sequence of row operations on B to obtain the following equivalent matrix.

$$C = \begin{bmatrix} 1 & 0 & e & f \\ 0 & 1 & g & h \end{bmatrix}.$$

Then

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = A^{-1}.$$

It should be pointed out that an equivalent matrix of the form C is possible only when A has an inverse.

The reader should note carefully how this rule follows from the preceding discussion.

Example 4. Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}.$$

Solution: Consider the matrix

$$B = \begin{bmatrix} 3 & -1 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix}.$$

We perform the following sequence of elementary row operations. (Note that such a sequence is not unique; there will be many possible sequences.)

$$B = \begin{bmatrix} 3 & -1 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Add } (-1) \times \text{ row (2) to row (1).}} \begin{bmatrix} 1 & -6 & 1 & -1 \\ 2 & 5 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-2) \times \text{ row (1) to row (2).}} \begin{bmatrix} 1 & -6 & 1 & -1 \\ 0 & 17 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply row (2) by } \frac{1}{17}.} \begin{bmatrix} 1 & -6 & 1 & -1 \\ 0 & 1 & \frac{-2}{17} & \frac{3}{17} \end{bmatrix}$$

$$\xrightarrow{\text{Add } 6 \times \text{ row (2) to row (1).}} \begin{bmatrix} 1 & 0 & \frac{5}{17} & \frac{1}{17} \\ 0 & 1 & \frac{-2}{17} & \frac{3}{17} \end{bmatrix}$$

We conclude that

$$A^{-1} = \begin{bmatrix} \frac{5}{17} & \frac{1}{17} \\ \frac{-2}{17} & \frac{3}{17} \end{bmatrix}.$$

Check:

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{17} & \frac{1}{17} \\ \frac{-2}{17} & \frac{3}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 5. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}.$$

Solution:

$$B = \begin{bmatrix} 2 & -3 & 1 & 0 \\ -6 & 9 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Add } 3 \times \text{ row (1) to row (2).}} \begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Now there is no sequence of row operations that will change the first two columns of this matrix into the desired form. We conclude that matrix A does not have an inverse. (Of course, we would recognize this immediately by checking that $ad - bc$ has the value $2(9) - (-3)(-6) = 0$ for this matrix.)

EXERCISE 8.7

Use the method of Section 8.7 to find inverses for the following matrices, where possible.

1. $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix}$
4. $\begin{bmatrix} 3 & -6 \\ 1 & 4 \end{bmatrix}$
5. $\begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix}$
6. $\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$
7. $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$
8. $\begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}$
9. $\begin{bmatrix} 6 & 0 \\ -3 & 2 \end{bmatrix}$

8.8. The Function $\mathbf{x} \longrightarrow A\mathbf{x}$

If A is a 2×2 matrix and $\mathbf{x}$ is a 2×1 matrix (a column vector with two entries), then the mapping

$$\mathbf{x} \to A\mathbf{x}$$

is a function whose domain is the set of all ordered pairs of real numbers written as column vectors. The range is either this same set or a subset thereof.

From the definition of matrix multiplication, we know that, for given A and given $\mathbf{x}$, the product matrix $A\mathbf{x}$ is a unique column vector. In the mapping $\mathbf{x} \to A\mathbf{x}$, there is a unique column vector $A\mathbf{x}$ corresponding to each column vector $\mathbf{x}$, and so this mapping is indeed a function.

Example 1. State the domain and range of the function $\mathbf{x} \to A\mathbf{x}$ if

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}.$$

Solution: Let $A\mathbf{x} = \mathbf{y}$ with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

corresponds to the system of equations, that is, to the transformation

$$A \quad \begin{cases} y_1 = 3x_1 - x_2 \\ y_2 = 2x_1 + x_2 \end{cases}.$$

We now know that this transformation, and its corresponding matrix, have inverses if and only if $ad - bc \neq 0$. In this case, $ad - bc$ has the value $3(1) - (-1)(2) = 5$;

hence, the transformation A does have an inverse. Again, we know that this means that A is a one-to-one mapping of the set of ordered pairs (x_1, x_2) onto the set of ordered pairs (y_1, y_2), or a one-to-one mapping of the set of column vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ onto the set of column vectors $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Thus

$$\mathbf{x} \rightarrow A\mathbf{x} = \mathbf{y}$$

is a function with domain and range the set

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in Re \right\},$$

that is, the set of all column vectors with two real components.

We have used the symbol V_2 to refer to the set of all row vectors with two components. We will use the symbol V_2' to refer to the set of all column vectors with two components;

$$V_2' = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in Re \right\}.$$

Thus, in Example 1, V_2' is both the domain and the range of the function $\mathbf{x} \rightarrow A\mathbf{x}$.

Example 2. Find the domain and range of the function $\mathbf{x} \rightarrow B\mathbf{x}$ if

$$B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}.$$

Solution: Let $B\mathbf{x} = \mathbf{y}$ as in Example 1. Then $B\mathbf{x} = \mathbf{y}$ corresponds to the transformation

$$B \quad \begin{cases} y_1 = 3x_1 - x_2 \\ y_2 = -6x_1 + 2x_2 \end{cases}.$$

Neither the transformation B nor the corresponding matrix has an inverse, since $ad - bc$ has value 0. B is a mapping of the set V_2' *into itself*. Indeed, the image vectors, that is, the elements of the domain, will all be of the form

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

with $b = -2a$.

The domain of the function $\mathbf{x} \rightarrow B\mathbf{x}$ is V_2' and the range is the subset

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in Re, b = -2a \right\}$$

of V_2'.

Example 3. Show that $A(2\mathbf{x}) = 2A\mathbf{x}$, if

$$A = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution:

$$2\mathbf{x} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}.$$

Hence

$$\begin{aligned} A(2\mathbf{x}) &= \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \\ &\quad \begin{bmatrix} 8x_1 + 6x_2 \\ -2x_1 + 4x_2 \end{bmatrix} \\ &= 2\begin{bmatrix} 4x_1 + 3x_2 \\ -x_1 + 2x_2 \end{bmatrix} \\ &= 2\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2A\mathbf{x}. \end{aligned}$$

This result is true in general. Restricting our attention to 2×2 real matrices A, we may say that

$$A(k\mathbf{x}) = kA\mathbf{x} \quad \text{if } k \in Re \text{ and } \mathbf{x} \in V_2'.$$

Example 4. Show that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ if

$$A = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Solution:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}.$$

Hence,

$$\begin{aligned} A(\mathbf{x} + \mathbf{y}) &= \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \\ &= \begin{bmatrix} 4(x_1 + y_1) + 3(x_2 + y_2) \\ -(x_1 + y_1) + 2(x_2 + y_2) \end{bmatrix} \\ &= \begin{bmatrix} (4x_1 + 3x_2) + (4y_1 + 3y_2) \\ (-x_1 + 2x_2) + (-y_1 + 2y_2) \end{bmatrix} \\ &= \begin{bmatrix} 4x_1 + 3x_2 \\ -x_1 + 2x_2 \end{bmatrix} + \begin{bmatrix} 4y_1 + 3y_2 \\ -y_1 + 2y_2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= A\mathbf{x} + A\mathbf{y}. \end{aligned}$$

This result is also true in general. Again restricting our attention to real matrices A, we may say that

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} \quad \text{if } \mathbf{x}, \mathbf{y} \in V_2'.$$

The properties

(i) $A(k\mathbf{x}) = kA\mathbf{x}$

(ii) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$

are called the *linear properties* of the function $\mathbf{x} \to A\mathbf{x}$. They may be combined in the single property

(iii) $A(k\mathbf{x} + l\mathbf{y}) = kA\mathbf{x} + lA\mathbf{y}, \quad k, l \in Re, \mathbf{x}, \mathbf{y} \in V_2'.$

EXERCISE 8.8

1. State the domain and range of the function $\mathbf{x} \to A\mathbf{x}$ for the following matrices A.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$ (h) $\begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$

2. For the matrices A of question (1), check that $A(-3\mathbf{b}) = (-3)A\mathbf{b}$ if

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

3. For the matrices A in question (1), check that $A(\mathbf{b}+\mathbf{c}) = A\mathbf{b} + A\mathbf{c}$ if

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

4. Prove that $A(k\mathbf{x}) = kA\mathbf{x}$ if $k \in Re$, $\mathbf{x} \in V_2'$, and A is a 2×2 real matrix.

5. Prove that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ if $\mathbf{x}, \mathbf{y} \in V_2'$ and A is a 2×2 real matrix.

6. Prove algebraically that if $A(k\mathbf{x}) = kA\mathbf{x}$ and $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, then $A(k\mathbf{x} + l\mathbf{y}) = kA\mathbf{x} + lA\mathbf{y}$ for $k, l \in Re$.

7. Prove that if $A(k\mathbf{x} + l\mathbf{y}) = kA\mathbf{x} + lA\mathbf{y}$ for $k, l \in Re$, then $A(k\mathbf{x}) = kA\mathbf{x}$ and $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

8.9. Some Special Transformations

In this section, we shall consider some special linear transformations and their corresponding matrices.

We have already encountered the identity and the zero transformations corresponding to the matrices I and O. We repeat them here for completeness.

(1) *The Identity Transformation and Identity Matrix*
The transformation

$$I \quad \begin{cases} u = x + 0y \\ v = 0x + y \end{cases}$$

or

$$I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

corresponds to the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(2) *Zero Transformation and Zero Matrix*
The transformation

$$O \quad \begin{cases} u = 0x + 0y \\ v = 0x + 0y \end{cases}$$

or

$$O \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

corresponds to the matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(3) *Rotations of the Plane*

We have also seen that the transformation

$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

given by the system of equations

$$\begin{cases} u = x \cos\theta - y \sin\theta \\ v = x \sin\theta + y \cos\theta \end{cases}$$

represents a rotation in the plane through angle θ. The corresponding matrix is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For this matrix, $ad - bc$ has the value $\cos^2\theta + \sin^2\theta = 1$, and so an inverse transformation and matrix exists. Solving for x and y in terms of u and v yields

$$\begin{cases} x = u \cos\theta + v \sin\theta \\ y = -u \sin\theta + v \cos\theta \end{cases}.$$

The inverse transformation should represent a rotation through angle $-\theta$. We check that this is so by noting that the inverse matrix is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = R_{-\theta}.$$

Example 1. Find the images of the points $P_1(0,0)$, $P_2(1,0)$, and $P_3(1,1)$ under the transformation effecting a rotation of 90°. Draw the triangle $P_1P_2P_3$ and the image triangle $P_1'P_2'P_3'$ on the same diagram.

Solution: The required transformation is $R_{\pi/2}$ with

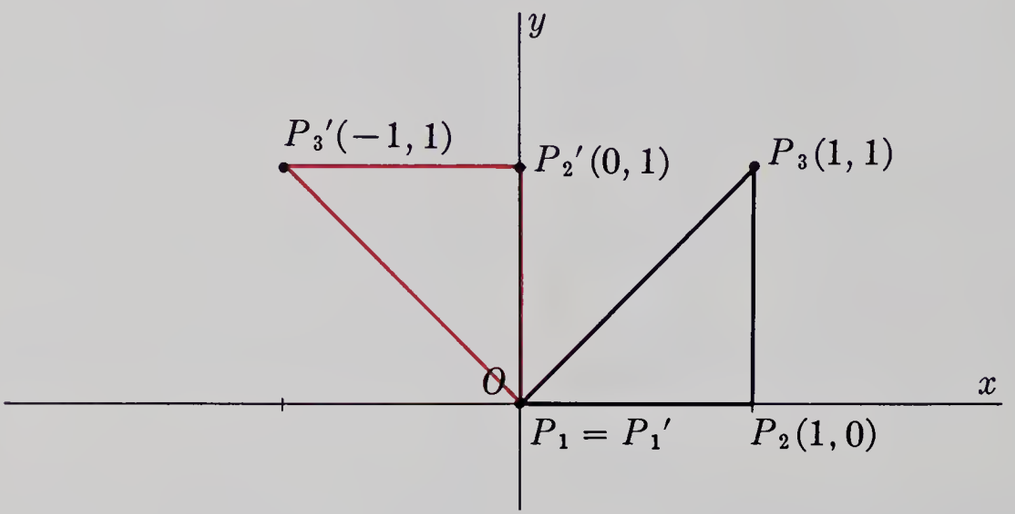
$$R_{\pi/2}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\frac{\pi}{2} - y\sin\frac{\pi}{2} \\ x\sin\frac{\pi}{2} + y\cos\frac{\pi}{2} \end{bmatrix};$$

that is,

$$R_{\pi/2}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Then

$$R_{\pi/2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_{\pi/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad R_{\pi/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$



Note that the lengths of line segments are invariant under such a rotation.

(4) *Uniform Magnification*

The transformation

$$M \quad \begin{cases} u = kx + 0y \\ v = 0x + ky \end{cases}$$

or

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

is called a *uniform magnification*. The images $M(P)$ and $M(Q)$ of two points P and Q determine a line segment whose length is kPQ. The corresponding matrix is

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

(5) *Magnification in the x-Direction*

$$M_x \quad \begin{cases} u = kx + 0y \\ v = 0x + y \end{cases}$$

or

$$M_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}.$$

(See question (5), Exercise 8.9.)

(6) *Magnification in the y-Direction*

$$M_y \quad \begin{cases} u = x + 0y \\ v = 0x + ky \end{cases}$$

or

$$M_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}.$$

(See question (6), Exercise 8.9.)

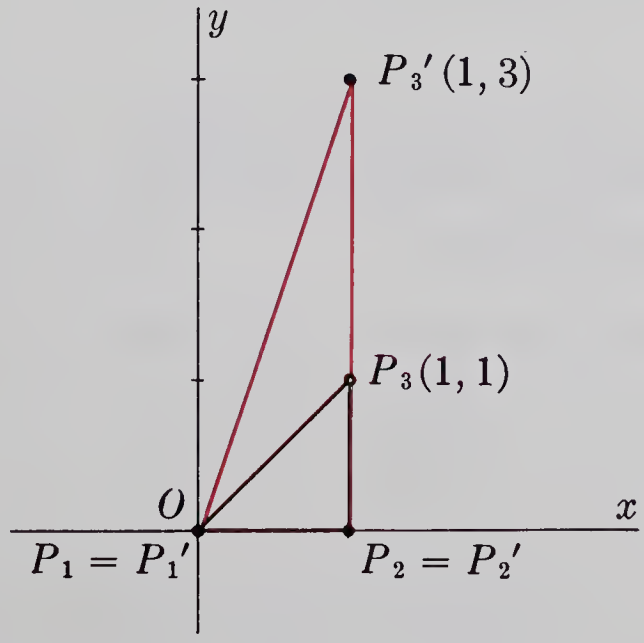
Example 2. Find the images of the points $P_1(0, 0)$, $P_2(1, 0)$, and $P_3(1, 1)$ under the transformation effecting a magnification in the y-direction (let $k = 3$). Show the triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$ on the same diagram.

Solution: The transformation involved is M_y with $k = 3$.

$$M_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3y \end{bmatrix}.$$

Then

$$M_y \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad M_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad M_y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$



(7) *Projection Onto the x-Axis*

$$P_x \quad \begin{cases} u = x + 0y \\ v = 0x + 0y \end{cases}$$

or

$$P_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

(8) *Projection Onto the y-Axis*

$$P_y \quad \begin{cases} u = 0x + 0y \\ v = 0x + y \end{cases}$$

or

$$P_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

(9) *Reflection in the y-Axis*

$$R_y \quad \begin{cases} u = -x + 0y \\ v = 0x + y \end{cases}$$

or

$$R_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

(See question (8), Exercise 8.9.)

(10) *Reflection in the x-Axis*

$$R_x \quad \begin{cases} u = x + 0y \\ v = 0x - y \end{cases}$$

or

$$R_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

(See question (8), Exercise 8.9.)

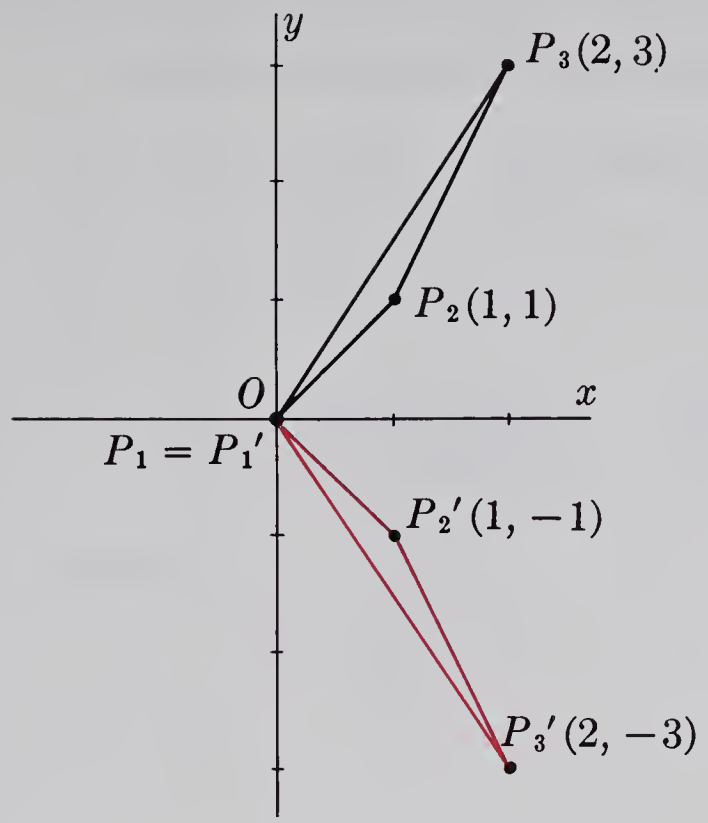
Example 3. Find the images of the points $P_1(0, 0)$, $P_2(1, 1)$, and $P_3(2, 3)$ under the transformation effecting a reflection in the x-axis. Show the triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$ on the same diagram.

Solution: The transformation involved is R_x with

$$R_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Then

$$R_x \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad R_x \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad R_x \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$



(11) *Reflection in the Origin*

$$R_O \begin{cases} u = -x + 0y \\ v = 0x - y \end{cases}$$

or

$$R_O \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

(See question (8), Exercise 8.9.)

(12) *Shear in the x-Direction*

$$S_x \begin{cases} u = x + ay \\ v = 0x + y \end{cases}$$

or

$$S_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}.$$

(See question (9), Exercise 8.9.)

(13) *Shear in the y-Direction*

$$S_y \begin{cases} u = x + 0y \\ v = ax + y \end{cases}$$

or

$$S_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ax + y \end{bmatrix}.$$

(See question (9), Exercise 8.9.)

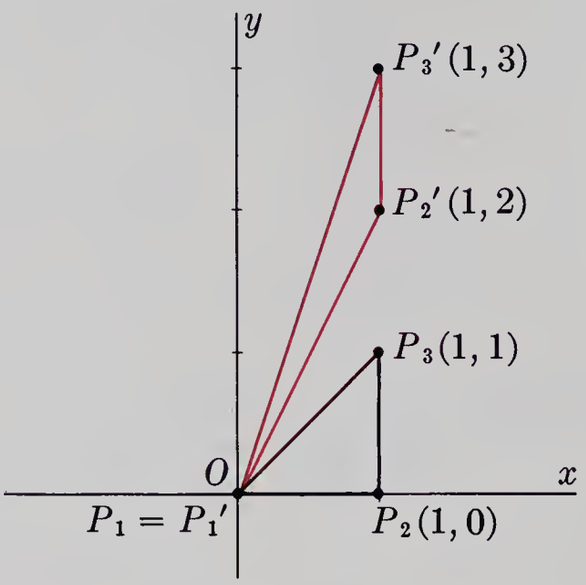
Example 4. Find the images of the points $P_1(0, 0)$, $P_2(1, 0)$, and $P_3(1, 1)$ under the transformation effecting a shear in the y-direction (let $a = 2$). Show the triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$ on the same diagram.

Solution: The transformation involved is S_y with $a = 2$.

$$S_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + 2x \end{bmatrix}.$$

Then

$$S_y \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad S_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad S_y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$



In addition to the preceding basic transformations, we have many combinations of them.

Example 5. Find the transformation equivalent to a shear in the x-direction followed by a reflection in the x-axis.

Solution: The transformation is the product R_xS_x.

$$(x, y) \xrightarrow{S_x} (x + ky, y) \xrightarrow{R_x} (x + ky, -y)$$

Thus R_xS_x is the transformation

$$R_xS_x \quad \begin{cases} u = x + ky \\ v = 0x - y \end{cases}$$

and the corresponding matrix is

$$\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}.$$

Example 6. Find the transformation equivalent to a magnification in the y-direction followed by a projection onto the y-axis.

Solution: The transformation is the product P_yM_y.

$$(x, y) \xrightarrow{M_y} (x, ky) \xrightarrow{P_y} (0, ky)$$

We also write

$$P_yM_y\begin{bmatrix} x \\ y \end{bmatrix} = P_y\left(M_y\begin{bmatrix} x \\ y \end{bmatrix}\right) = P_y\begin{bmatrix} x \\ ky \end{bmatrix} = \begin{bmatrix} 0x + 0y \\ 0x + ky \end{bmatrix}.$$

Thus P_yM_y is the transformation

$$P_yM_y \quad \begin{cases} u = 0x + 0y \\ v = 0x + ky \end{cases}$$

and the corresponding matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}.$$

EXERCISE 8.9

1. Write matrices corresponding to rotations of
   (a) 0°, (b) 30°, (c) 45°, (d) 60°,
   (e) 90°, (f) 120°, (g) 180°, (h) 270°.

2. (a) If $R_{\pi/6}$ is the matrix in question (1b) and $R_{\pi/3}$ is the matrix in question (1d), show that
   $$R^2_{\pi/6} = R_{\pi/3}.$$
   (b) Show that
   $$R^2_{\pi/4} = R_{\pi/2}.$$
   (c) Show that
   $$R^3_{\pi/3} = R_{\pi}.$$

3. Find the images $T(P)$ of the points $P_1(0, 2)$, $P_2(-2, -1)$, $P_3(2, -1)$ under the transformation T effecting a rotation of 60°. Draw the triangles $P_1P_2P_3$ and $T(P_1)T(P_2)T(P_3)$ on the same diagram.

4. Repeat question (3) if T is the transformation effecting a uniform magnification (let $k = 2$). Compare the areas of the resulting triangles.

5. (a) Write the matrix corresponding to a magnification in the x-direction with
   (i) $k = 2$, (ii) $k = -2$.
   (b) Find the images $T(P)$ of the points $P_1(1, 2)$, $P_2(3, 2)$, and $P_3(5, 7)$ under the transformation T effecting a magnification in the x-direction (let $k = 2$). Draw the triangles $P_1P_2P_3$ and $T(P_1)T(P_2)T(P_3)$ and compare their areas.
   (c) Repeat part (b) using $k = -3$.

6. Repeat question (5) using the points $P_1(1, -3)$, $P_2(1, 5)$, and $P_3(2, 8)$ and their images under the transformation effecting a magnification in the y-direction with (a) $k = 3$, (b) $k = -1$.

7. Write matrices corresponding to
   (a) a projection onto the x-axis, (b) a projection onto the y-axis.

8. (a) Write the matrices corresponding to a reflection in
   (i) the y-axis, (ii) the x-axis, (iii) the origin.
   (b) Find the images of the points $P_1(1, 3)$, $P_2(2, 5)$, $P_3(3, 7)$, under the transformation T effecting a reflection in the y-axis. Draw the two triangles involved.
   (c) Repeat part (b) if T is a reflection in the x-axis.
   (d) Repeat part (b) if T is a reflection in the origin.

9. (a) Write the matrix corresponding to
   (i) a shear in the x-direction, (ii) a shear in the y-direction.
   (b) Find the images of the points $P_1(0, 0)$, $P_2(3, 0)$, $P_3(1, 4)$, under the shear
   $$T \begin{cases} u = x + 3y \\ v = y \end{cases}.$$
   Draw the triangles determined by the two sets of points.
   (c) Repeat part (b) for the shear
   $$T \begin{cases} u = x \\ v = -2x + y \end{cases}.$$

10. Find the transformation equivalent to a reflection in the x-axis followed by a magnification in the y-direction. State the matrix of the product transformation.

11. Find the transformation equivalent to a uniform magnification followed by a shear in the x-direction followed by projection onto the x-axis. State the matrix of the product transformation.

12. Find the transformation equivalent to a magnification in the y-direction followed by reflection in the origin followed by a shear in the x-direction. State the matrix of the product transformation.

13. Using the addition formulae for sine and cosine, show that
   $$R_\theta R_\phi = R_{\theta + \phi}.$$
   Interpret this result geometrically.

14. Use mathematical induction to show that
   $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$
   Give a geometric interpretation of this result.

Chapter Summary

Multiplication of matrices defined in terms of the "multiplication" of the corresponding linear transformations

Rule for multiplication of matrices

Properties of matrix multiplication: (i) noncommutativity in general, (ii) associativity, (iii) existence of identity, (iv) distributivity

Divisors of zero

The inverse of a matrix

The linear matrix equation $AX = B$

The solution of the linear matrix equation $AX = B$

The inverse of a matrix using elementary row operations

The function $\mathbf{x} \rightarrow A\mathbf{x}\cdot$ domain and range

Some special transformations: identity, zero, rotation, uniform magnification, magnification in the x- or y-direction, reflection in the x- or y-axis or the origin, projection onto the x- or y-axis, shear in the x- or y-direction

REVIEW EXERCISE 8

1. Write linear transformations A and B corresponding to the matrices
$$A = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}$$
in such a way that the product transformation BA can be formed and hence the product BA of matrices B and A can be obtained.

2. Rework question (1) in such a way as to obtain AB.

3. Prove that AB does not always equal BA if A and B are 2×2 real matrices.

4. If
$$A = \begin{bmatrix} 3 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 \\ 2 & -1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix},$$
find
(a) AB, (b) BC, (c) CA, (d) $A^2 + B^2 + C^2$.

5. If A is a 3×2 matrix, is it possible to form AB if B is
(a) 3×2? (b) 2×3? (c) 3×3? (d) 2×2?

6. Show that

(a) $A(BC) = (AB)C$ (b) $A(B + C) = AB + AC$

for the matrices of question (4).

7. Find AB if

$$A = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

What multiplicative property does this illustrate?

8. Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 6 \end{bmatrix}$$

by finding the inverse of the corresponding transformation.

9. Given

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix},$$

find

(a) A^{-1}, (b) B^{-1}, (c) AB, (d) $(AB)^{-1}$.

Show that $(AB)^{-1} = B^{-1}A^{-1}$.

10. Given

$$A = \begin{bmatrix} 4 & 4 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ -1 & 1 \end{bmatrix},$$

find

(a) AB, (b) $(AB)^{-1}$, (c) B^{-1}.

Show that $(AB)^{-1} A = B^{-1}$.

11. (a) If

$$A = \begin{bmatrix} 4 & 4 \\ -1 & 3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

find

(i) $A\mathbf{r}$, (ii) $A\mathbf{s}$, (iii) $A\mathbf{t}$.

(b) Explain why it is impossible to find $\mathbf{r}A$.

12. Write the system

$$\begin{aligned} 3x - 2y &= 4 \\ x + 5y &= -3 \end{aligned}$$

as a single linear matrix equation.

13. Show that

(a) $A(3\mathbf{r}) = 3A\mathbf{r}$

(b) $A(\mathbf{r} + \mathbf{s}) = A\mathbf{r} + A\mathbf{s}$

(c) $A(2\mathbf{r} + 3\mathbf{s} - \mathbf{t}) = 2A\mathbf{r} + 3A\mathbf{s} - A\mathbf{t}$

for the matrices of question (11).

14. Express the following systems as linear matrix equations of the form $AX = B$; find A^{-1} if it exists and use it to solve the systems.

(a) $\begin{aligned} 4x + y &= 0 \\ -x + 3y &= 2 \end{aligned}$ (b) $\begin{aligned} 4x - y &= 3 \\ 2x + 4y &= -2 \end{aligned}$ (c) $\begin{aligned} 3x - y &= 2 \\ 3x + y &= 6 \end{aligned}$

15. State the domain and range of the function $\mathbf{x} \to A\mathbf{x}$ if

(a) $A = \begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}$.

16. Let T be the transformation

$$T \quad \begin{cases} u = x + 3y \\ v = 0x + y \end{cases}$$

effecting a shear in the x-direction. Plot the points $P_1(3, 0)$, $P_2(0, 3)$, and $P_3(3, 3)$ and their images under T and draw the two triangles thus formed.

17. Let R_x be the reflection in the x-axis, R_y be the reflection in the y-axis, and R_O be the reflection in the origin. Show that

$$R_x R_y = R_y R_x = R_O .$$

State the corresponding matrices and show the corresponding result for these matrices.

18. Find the single transformation equivalent to a rotation of 45° followed by a reflection in the origin. Show that this transformation is equivalent to a rotation of 225°.

ANSWERS

Chapter 1

EXERCISE 1.1 (page 4)

2. (a) Infinitely many. (b) 1. (c) None.

3. (a) Infinitely many. (b) Infinitely many. (c) One if the points are non-collinear, infinitely many otherwise.

4. 90°. **5.** 17 in. **6.** 14 in., 67°, 34°.

EXERCISE 1.3 (page 12)

1. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{BC} = \overrightarrow{AD}$.

2. $\mathbf{u} + \mathbf{v} = \overrightarrow{QS} = \mathbf{v} + \mathbf{u}$, $\mathbf{u} - \mathbf{v} = \overrightarrow{RP}$, $\mathbf{v} - \mathbf{u} = \overrightarrow{PR}$.

3. (a) $\overrightarrow{RV}$. (b) $\overrightarrow{RV}$. (c) $\overrightarrow{RT}$. (d) $\overrightarrow{RU}$. (e) $\overrightarrow{PS}$. (f) $\overrightarrow{RU}$. (g) $\overrightarrow{RU}$. (h) $\overrightarrow{PS}$.

4. **0**.

8. (a) $\overrightarrow{DE}$. (b) None. (c) $\overrightarrow{CD}$. (d) $\overrightarrow{AC}$. (e) $\overrightarrow{EB}$. (f) $\overrightarrow{AE}$. (g) $\overrightarrow{AC}$. (h) $\overrightarrow{AC}$.

9. 15, 16°. **10.** 11, 52°. **11.** 13, 13°. **12.** 10.

EXERCISE 1.4 (page 16)

6. (a) $\overrightarrow{RT}$. (b) $\overrightarrow{RW}$.

7. (a) $\overrightarrow{PS}$. (b) **0**. (c) $\overrightarrow{QR}$. (d) $\overrightarrow{RQ}$.

EXERCISE 1.5 (page 21)

1. (a) 50 lb. (b) 69 lb. (c) 70 lb. (d) 21 lb. (e) 5 lb. (f) 142 lb.

2. 44 lb., 7°. **3.** 28 lb., −82°. **4.** 28 lb. **5.** 97 lb.

6. 30 lb., 78°. **7.** 17 lb., −150° to the horizontal.

8. 13 lb., −154° to the horizontal. **9.** 449 m.p.h., 56°. **10.** 670 m.p.h., 63°.

11. 21 knots, 337°. **12.** 18° to right of destination, 12 m.p.h.

13. 167°, 65 min. **14.** 40 lb., 49°. **15.** 40 lb., 49°.

EXERCISE 1.6 (page 25)

1. (a) 35 lb., 20 lb. (b) 42 lb., 42 lb. (c) 100 lb., 173 lb.
(d) 16 lb., 11 lb. (e) 42 lb., 58 lb. (f) 42 lb., 15 lb.

2. 21 lb., 28 lb.; 37°. **3.** 14 lb., 14 lb. **4.** 172 lb.
5. 42 lb. **6.** 785 lb. **7.** 34°, 40°. **8.** 35 lb.
9. 18 lb. **10.** 9,110 ft. per min.; 386 m.p.h.
11. 762,000 ft. per min.; 5000 m.p.h. **12.** 12,200 ft. per min.; 380 m.p.h.

EXERCISE 1.7 (page 30)

4. $\overrightarrow{PS} = 3\mathbf{v} - \mathbf{u}$, $\overrightarrow{RS} = -3\mathbf{u}$. **5.** $\overrightarrow{BD} = 2\mathbf{u} + \mathbf{v}$, $\overrightarrow{BC} = 2\mathbf{u} - \mathbf{v}$.
9. $2\mathbf{u} + \mathbf{v} + 5\mathbf{w}$; $\sqrt{41}$, $5\sqrt{17}$, 21.

EXERCISE 1.8 (page 39)

1. $\mathbf{w} = \frac{3\sqrt{3}}{5}\mathbf{u} + \frac{1}{2}\mathbf{v}$; $\mathbf{v} = 2\mathbf{w} - \frac{6\sqrt{3}}{5}\mathbf{u}$. **2.** 2.30, 1.54.
3. $\mathbf{w} = 2.88\mathbf{u} + 1.44\mathbf{v}$. **4.** $\overrightarrow{OB} = \frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OC}$. **5.** $\mathbf{v} = \frac{4}{3}\mathbf{u} - \frac{1}{3}\mathbf{w}$.

REVIEW EXERCISE 1 (page 42)

1. (a) $\overrightarrow{PR}$. (b) $\overrightarrow{PT}$. (c) $\overrightarrow{QT}$. (d) $\overrightarrow{SP}$. (e) $\overrightarrow{PT}$. (f) $\overrightarrow{SR}$.
2. (a) $\overrightarrow{AG}$. (b) $\overrightarrow{EG}$. (c) $\overrightarrow{AC}$. (d) $\overrightarrow{BH}$. (e) $\overrightarrow{FC}$. (f) $\overrightarrow{CF}$.
(g) $\overrightarrow{AF}$. (h) **0**. **3.** 6, 27°. **4.** $2\mathbf{u}$.
5. 87 pounds, 37° from the 40-pound force.
6. 61 gm., 141° from the 80-gram force.
7. 39 lb. **8.** 187 m.p.h., on a bearing of 135°. **9.** 203°, 20 knots.
10. 26 lb., 31 lb. **11.** 25 lb., 60 lb.; 23°.
12. 3,612 ft. per min.; 113 m.p.h. **13.** $\mathbf{v} = \frac{3\sqrt{3}}{5}\mathbf{u} + \frac{3}{8}\mathbf{w}$.
14. $\mathbf{u} = -3\mathbf{v} - \frac{3\sqrt{3}}{5}\mathbf{w}$. **18.** $\mathbf{v} = \frac{2}{3}\mathbf{w} + \frac{1}{3}\mathbf{u}$. **19.** $\overrightarrow{OQ} = \frac{5}{3}\overrightarrow{OP} - \frac{2}{3}\overrightarrow{OR}$.

Chapter 2

EXERCISE 2.1 (page 47)

2. All co-ordinates are positive in the first octant. By the signs $(+\,+\,+)$, $(+\,+\,-)$, $(+\,-\,+)$, $(+\,-\,-)$, $(-\,+\,+)$, $(-\,+\,-)$, $(-\,-\,+)$, $(-\,-\,-)$.
3. (a) $x = 0$. (b) $y = 0$. (c) $z = 0$.
4. (a) $y = z = 0$. (b) $x = z = 0$. (c) $x = y = 0$.
5. (a) $y = 0, z = 0$. (b) $x = 0, z = 0$. (c) $x = 0, y = 0$.

EXERCISE 2.2 (page 49)

1. (a) $(6, 9)$. (b) $(-2, 3)$. (c) $(4, 7)$. (d) $(5, -1, 5)$.
(e) $(0, -2, -4)$. (f) $(4, 0, -5)$. (g) $(10, 40)$. (h) $(-12, -21)$.
(i) $(0, 0)$. (j) $(-6, -3, 12)$. (k) $(8, 2, -6)$. (l) $(2, 1, -3)$.
(m) $(0, 0)$. (n) $(1, 5)$. (o) $(1, 10)$. (p) $(2, 0, -2)$.
(q) $(-7, 2, 4)$. (r) $(-5, 0, 6)$.

4. $(0, 15, 33)$. **5.** $a = 2, b = 1$. **6.** $a = -5, b = -\frac{1}{3}, c = -20$.

7. $x = -\frac{23}{2}, y = 1, z = \frac{32}{3}$. **8.** $p = -\frac{7}{3}, q = \frac{4}{3}, r = 2$.

EXERCISE 2.3 (page 52)

3. (a) $(2, 5)$. (b) $(-3, 1)$. (c) $(-5, -2)$. (d) $(-2, -5)$.
(e) $(3, -1)$. (f) $(5, 2)$. (g) $(2, 3)$. (h) $(-5, 9)$.
(i) $(-2, -4)$. (j) $(-3, -5)$.

4. (a) $(-2, -2)$. (b) $(-11, 0)$. (c) $(8, 5)$. (d) $(-6, 1)$.

5. (a) $(3, -1, 4)$. (b) $(-3, -2, -4)$. (c) $(2, 1, 1)$.
(d) $(4, -1, 0)$. (e) $(9, -2, 1)$. (f) $(0, -8, 0)$.

6. (a) $(-6, 1, 0)$. (b) $(2, 5, 5)$. (c) $(-2, -2, 4)$. (d) $(7, -2, -3)$.

7. The components of one are the additive inverses of the corresponding components of the other.

8. Equal in magnitude but opposite in direction. **9.** The null vector.

10. (a) $b = c = 0$. (b) $a = c = 0$. (c) $a = b = 0$.
(b) $a = 0$. (e) $b = 0$. (f) $c = 0$.

EXERCISE 2.4 (page 56)

1. (a) $(9, 3)$. (b) $(-2, 6)$. (c) $(-1, -2)$. (d) $(2, -6)$. (e) $(3, 0)$. (f) $(-7, 2)$.

2. (a) $(4, 9)$. (b) $(2, 4)$. (c) $(2, 0)$. (d) $(-4, -2)$.

3. (a) $(6, 0)$. (b) $(-4, -7)$.

7. (a) $(9, 5)$. (b) $(4, 1)$. (c) $(-5, 9)$. (d) $(-7, -\frac{9}{2})$.

EXERCISE 2.5 (page 68)

1. For $k \in Re$ (a) $(k, 3k)$. (b) $(-3k, 4k)$. (c) $(3k, 5k, -2k)$.
(d) $(-6k, -k, 5k)$. (e) $(k, -2k, 0)$. (f) $(-k, 2k, 3k)$.

2. (a) $(3k, 4k)$. (b) $(8k, -3k)$. (c) $(-7k, 2k)$.
(d) $(-5k, 2k)$. (e) $(-5k, -2k, 4k)$. (f) $(4k, -3k, 2k)$.

3. (a) Yes. (b) No. (c) Yes. (d) No. (e) Yes. (f) No.

4. (a) -4. (b) -2. (c) -10.
(d) $k = -\frac{1}{8}, l = \frac{1}{8}$. (e) $k = 9, l = 12$. (f) $k = 0, l = -8$.

EXERCISE 2.6 (page 61)

1. For $k, l \in Re$ (a) $(k, 2l, 0)$. (b) $(k + 2l, 2k, -l)$.
(c) $(3k - 2l, 2k + 5l, 2k - 3l)$. (d) $(-3k + l, 2k, 4k + 3l)$.
(e) $(-2k, -2k - 3l, 5k + 4l)$. (f) $(-4k + 3l, -l, 5l)$.

2. (a) 5. (b) 0. (c) No solution. (d) 1.

3. (a) Yes. (b) Yes. (c) Yes. (d) Yes. (e) Yes. (f) Yes. (g) No.
(h) Yes. (i) No. (j) Yes. (k) Yes. (l) Yes. (m) No. (n) Yes.

4. (a) Yes. (b) Yes. (c) No. (d) Yes. **5.** No.

6. (a) No. (b) No. (c) No. (d) No.

EXERCISE 2.7 (page 65)

1. (a) $(0, 0) = 0(3, 5) + 0(2, -3)$. (b) $(3, -4) = 3(1, 0) - 4(0, 1)$.
(c) $(-3, 3) = 0(4, 7) - 3(1, -1)$. (d) $(4, 4) = -4(-1, -1) + 0(-3, -3)$.
(e) $(0, 0) = 0(3, 7) + 0(2, 9)$. (f) $(4, 2) = 0(5, 9) + 2(2, 1)$.
(g) $(3, 2) = (1, 0) + 2(1, 1)$. (h) $(5, 2) = \frac{5}{3}(3, -2) + \frac{16}{3}(0, 1)$.
(i) $(5, 7) = \frac{11}{5}(-1, 1) + \frac{12}{5}(3, 2)$. (j) $(3, 1) = -\frac{17}{12}(-2, -1) + \frac{1}{12}(2, -5)$.

2. (a) No. (b) Yes; $(5, -2) = 5(1, 0) - \frac{2}{3}(0, 3)$.
(c) No. (d) Yes; $(5, -2) = -\frac{5}{2}(-2, 0) + \frac{2}{3}(0, -3)$.
(e) Yes; $(5, -2) = \frac{13}{6}(1, 3) + \frac{17}{12}(2, -6)$.
(f) Yes; $(5, -2) = \frac{13}{2}(2, -4) + 8(-1, 3)$.
(g) Yes; $(5, -2) = 0(3, -1) + 1(5, -2)$.
(h) Yes; $(5, -2) = \frac{9}{2}(2, 0) - 1(4, 2)$. (i) No.

3. (a) $(3, 2)$. (b) $(-1, 4)$. (c) $(6, 0)$. (d) $(-1, -2)$.

EXERCISE 2.8 (page 67)

1. (a) No. (b) Yes. (c) No. (d) No.

2. (a) Collinear.
(b) Not collinear; coplanar; $(-3, 3, 0) = -3(1, -1, 0) + 0(-1, 1, 1)$.
(c) Not collinear; coplanar; $(1, 1, 0) = (1, 0, 0) + (0, 1, 0)$.
(d) Not collinear; not coplanar. (e) Not collinear; not coplanar.
(f) Not collinear: coplanar; $(0, 4, 1) = 2(0, 2, \frac{1}{2}) + 0(0, -16, -18)$.
(g) Not collinear; not coplanar. (h) Not collinear; not coplanar.

3. (a) Collinear.
(b) Coplanar; $(-12, 4, 0) = 4(-3, 1, 0) + 0(2, 1, 4)$;
$(6, -2, 0) = -2(-3, 1, 0) + 0(2, 1, 4)$.
(c) Coplanar; $(-4, -6, 2) = -2(2, 3, -1) + 0(5, 1, 2)$;
$(-10, -2, -4) = 0(2, 3, -1) - 2(5, 1, 2)$.
(d) Coplanar; $(1, 2, 3) = (1, 0, 3) + (0, 2, 0)$;
$(-2, 0, -6) = -2(1, 0, 3) + 0(0, 2, 0)$.
(e) Noncoplanar. (f) Noncoplanar.
(g) Coplanar; $(4, 0, 3) = (2, 2, -1) + 2(1, -1, 2)$;
$(3, 1, 1) = (2, 2, -1) + (1, -1, 2)$.

EXERCISE 2.9 (page 70)

1. (a) Yes. (b) Yes. (c) Yes. (d) No. (e) No. (f) Yes.

2. (a) $(0, 1, 0) + (1, 0, 0) + \frac{1}{2}(0, 0, 2)$.
(b) $-1(-1, 0, 0) + \frac{1}{3}(0, 3, 0) - \frac{1}{2}(0, 0, -2)$.
(c) $-1(-1, 0, 1) + (0, 1, 0) + 2(0, 0, 1)$.
(f) $(3, -1, 1) - \frac{4}{3}(2, -1, 1) + \frac{2}{3}(1, 1, 2)$.

3. (a) $(3, -2, 1)$. (b) $(-4, 0, 2)$. (c) $(5, -3, -1)$.

4. (a) $(-1, 2, 1) = -1(1, 0, 0) + (0, 2, 0) + \frac{1}{3}(0, 0, 3)$.
(b) $(2, 2, 1) = \frac{17}{28}(3, 1, 0) + \frac{13}{28}(0, 3, 1) + \frac{5}{28}(1, 0, 3)$. (c) No solution.
(d) $(1, 1, 4) = 4(4, -1, 1) - 5(3, 0, -1) + 5(0, 1, -1)$.
(e) $(5, 2, 1) = (3, -3, 1) + \frac{1}{6}(2, 0, 0) + \frac{5}{3}(1, 3, 0)$.

REVIEW EXERCISE 2 (page 71)

1 (a) $(-3, 2)$. (b) $(15, -6, -21)$. (c) $(0, 18)$.
(d) $(-8, 7, -12)$. (e) $(3, -5, 6)$. (f) $(6, -4, -5)$.

2. (a) $(-7, 7)$. (b) $(5, -2)$. (c) $(7, -2, 7)$. (d) $(4, -1, 11)$.

3. (a) $(1, 9)$. (b) $(2, 6)$. **4.** (a) -2. (b) $k = -\frac{1}{3}, l = -\frac{2}{3}$.

5. (a) Yes. (b) No. (c) No. (d) No.

6. (a) Yes. (b) No. (c) No. (d) No.

7. (a) No. (b) Yes; $(4, -3) = -\frac{10}{7}(-1, 3) + \frac{9}{7}(2, 1)$.
(c) No. (d) Yes; $(4, -3) = -\frac{1}{3}(3, -1) + \frac{5}{3}(3, -2)$.
(e) Yes; $(4, -3) = 2(2, 0) - \frac{3}{5}(0, 5)$. (f) Yes; $(4, -3) = 18(1, 1) - 7(2, 3)$.

8. (a) Noncollinear, coplanar. (b) Collinear.
(c) Noncollinear, noncoplanar. (d) Noncollinear, noncoplanar.

9. (a) Coplanar. (b) Noncoplanar. (c) Noncoplanar.

10. (a) Yes; $(5, 1, -2) = \frac{5}{3}(3, 0, 0) + \frac{1}{2}(0, 2, 0) + (0, 0, -2)$.
(b) Yes; $(2, 2, -5) = \frac{7}{6}(3, -1, 0) + \frac{3}{2}(-1, 0, 3) - \frac{19}{6}(0, -1, 3)$.
(c) Yes; $(3, 2, 3) = \frac{9}{2}(4, -1, 2) - 2(1, 0, 3) - \frac{13}{2}(2, -1, 0)$.

Chapter 3

EXERCISE 3.1 (page 76)

1. 3; 1; 2; 5; 0; 3; 2; 2; 5; 6. **3.** 6; $2\sqrt{5}$; $\sqrt{37}$; $2\sqrt{26}$; $3\sqrt{2}$; 5; $3\sqrt{10}$; 0.

9. (a) Noncollinear. (b) Noncollinear. (c) Noncollinear.

10. (a) Right-angled. (b) Not right-angled. (c) Not right-angled.

EXERCISE 3.2 (page 78)

1. $3\sqrt{2}$; $\sqrt{29}$; $\sqrt{61}$; $\sqrt{29}$; 0; $\sqrt{74}$; $2\sqrt{2}$; $\sqrt{61}$; $\sqrt{29}$; $\sqrt{122}$.

5. (a) Noncollinear. (b) Noncollinear. (c) Noncollinear.

6. (a) Right-angled. (b) Right-angled. (c) Not right-angled.

EXERCISE 3.3 (page 81)

1. (a) 2. (b) 3. (c) 6. (d) 4. (e) $\sqrt{26}$. (f) $\sqrt{6}$.
(g) 5. (h) $3\sqrt{3}$. (i) $2\sqrt{10}$. (j) $\sqrt{53}$. (k) $\sqrt{41}$. (l) 3.
(m) $2\sqrt{5}$. (n) $\sqrt{11}$. (o) $\sqrt{13}$. (p) $\sqrt{14}$.

2. (a) $\sqrt{13}$. (b) $\sqrt{65}$. (c) $\sqrt{13}$. (d) $\sqrt{26}$. (e) $\sqrt{53}$. (f) $\sqrt{65}$.

3. (a) $\sqrt{26}$. (b) $\sqrt{6}$. (c) $\sqrt{21}$. (d) $5\sqrt{2}$. (e) $3\sqrt{3}$. (f) $\sqrt{77}$.
(g) $\sqrt{101}$. (h) $5\sqrt{2}$. (i) $3\sqrt{10}$.

EXERCISE 3.4 (page 83)

1. (a) $\pm\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. (b) $\pm\frac{1}{\sqrt{10}}(-3, 1)$. (c) $\pm\frac{1}{\sqrt{29}}(2, 5)$.
(d) $\pm\sqrt{2}(\frac{1}{2}, -\frac{1}{2})$. (e) $\pm\frac{1}{\sqrt{6}}(1, 1, 2)$. (f) $\pm\frac{1}{\sqrt{11}}(-1, 3, 1)$.
(g) $\pm\frac{1}{\sqrt{17}}(2, -2, 3)$. (h) $\pm\frac{1}{\sqrt{34}}(3, 3, -4)$.

2. (a) $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. (b) $\pm\left(\frac{4}{\sqrt{52}}, \frac{-6}{\sqrt{52}}\right)$. (c) $\pm\left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$.
(d) $\pm\left(\frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$ (e) $\pm\left(\frac{4}{\sqrt{66}}, \frac{-1}{\sqrt{66}}, \frac{-7}{\sqrt{66}}\right)$.
(f) $\pm\left(\frac{7}{\sqrt{58}}, 0, \frac{3}{\sqrt{58}}\right)$.

3. (a) $\pm\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$. (b) $\pm\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$. (c) $\pm\frac{1}{\sqrt{126}}(10, -5, -1)$.
(d) $\pm\frac{1}{\sqrt{46}}(6, -3, 1)$. (e) $\pm\frac{1}{\sqrt{86}}(-2, 1, 9)$. (f) $\pm\frac{1}{\sqrt{6}}(2, -1, 1)$.
(g) $\pm\frac{1}{\sqrt{2206}}(-42, 21, 1)$.

EXERCISE 3.5 (page 84)

1. (a) 3. (b) 5. (c) -2. (d) 17. (e) $\frac{3}{2}$. (f) $2ab$. (g) 0.
(h) 2. (i) 0. (j) 0. (k) -1. (l) 20. (m) -5. (n) 1.
(o) $3ab$. (p) $-a^2$.

2. (a) $\frac{15}{4}$. (b) $-\frac{19}{2}$. (c) 8. (d) $-\frac{3}{2}$.

3. (a) 1. (b) 1. (c) 1.

4. (a) 0. (b) 0. (c) 0.

5. (a) 25. (b) 14. (c) $a^2 + b^2 + c^2$.

6. (a) 14. (b) 14. (c) -3. (d) -3. (e) 78.
(f) 78. (g) -348. (h) -348. (i) 312.

7. (a) -3. (b) -3. (c) -21. (d) -21. (e) -54.
(f) -54. (g) -36. (h) -36. (i) -216.

EXERCISE 3.7 (page 89)

7. (a) $2|\mathbf{u}|^2 + 5\mathbf{u}\cdot\mathbf{v} - 12|\mathbf{v}|^2$. (b) $-6|\mathbf{u}|^2 - 12|\mathbf{v}|^2 + 17\mathbf{u}\cdot\mathbf{v}$.

8. (a) -130. (b) -234.

EXERCISE 3.9 (page 94)

1. (a) $\frac{17}{5\sqrt{26}}$. (b) $\frac{-12}{\sqrt{145}}$. (c) $\frac{-4}{\sqrt{17}}$.
(d) $\frac{9}{\sqrt{231}}$. (e) $\frac{12}{5\sqrt{10}}$. (f) $\frac{-22}{3\sqrt{114}}$.

2. (a) 122°. (b) 108°. (c) 23°. (d) 99°. (e) 80°. (f) 110°.

3. (a) 51°. (b) 9°. (c) 112°. (d) 97°. (e) 141°. (f) 0°.

4. (a) 135°. (b) 81°. (c) 72°.

5. $\angle P = 75°$, $\angle Q = 38°$, $\angle R = 67°$. **6.** $\angle P = 23°$, $\angle Q = 13°$, $\angle R = 144°$.

7. (a) $\frac{10}{21}$. (b) $\frac{1}{2}$. (c) -23. (d) $-\frac{21}{2}$. (e) 22. (f) $\frac{11}{12}$.

8. Yes, $\angle PQR = 90°$. **9.** No. **10.** $h = -4$, $k = \frac{7}{2}$.

EXERCISE 3.10 (page 98)

1. (a) $\frac{6}{\sqrt{13}}$. (b) $\frac{11}{5}$. (c) 0. (d) $\frac{1}{\sqrt{26}}$. (e) $\frac{3}{\sqrt{10}}$. (f) $\frac{7}{\sqrt{5}}$.
(g) $\frac{11}{\sqrt{14}}$. (h) $\frac{8}{\sqrt{14}}$. (i) 0. (j) $\frac{13}{\sqrt{21}}$. (k) $\frac{12}{\sqrt{14}}$. (l) $\frac{1}{\sqrt{14}}$.

2. (a) $\frac{35}{\sqrt{26}}$. $\frac{41}{\sqrt{33}}$. **3.** When $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

4. (a) No. (b) Yes; when $|\mathbf{u}| = |\mathbf{v}|$.

EXERCISE 3.11 (page 101)

1. (a) $(5, 0)$, $(0, \frac{1}{2})$. (b) $(-4, 0)$, $(0, 7)$. (c) $(6, 0)$, $(0, -2)$.
(d) $(-3, 0)$, $(0, -4)$.

2. $\sqrt{29}$. **3.** (a) $2\sqrt{5}$. (b) $\sqrt{17}$. **4.** $\sqrt{5}$.

5. $(3, 0)$, $(0, -5)$. **6.** $(3, 0)$, $(0, -5)$.

7. $\mathbf{w}_1 = (-\frac{7}{5}, -\frac{14}{5})$, $\mathbf{w}_2 = (\frac{22}{5}, -\frac{11}{5})$. **8.** $(-\frac{11}{5}, \frac{22}{5})$, $(-\frac{4}{5}, -\frac{2}{5})$.

9. $(\frac{51}{13}, \frac{-34}{13})$, $(\frac{14}{13}, \frac{21}{13})$.

10. Uphill force is 11 pounds. Lift force is 11 pounds.

11. Uphill force is 2050 pounds. Lift force is 900 pounds.

12. (a) $(-1, 0, 0)$, $(0, 5, 0)$, $(0, 0, 3)$. (b) $(6, 0, 0)$, $(0, -4, 0)$, $(0, 0, 2)$.
(c) $(-1, 0, 0)$, $(0, -6, 0)$, $(0, 0, 5)$.

13. $|\mathbf{F}_x| = 3$ lb.; $|\mathbf{F}_y| = 12$ lb.; $|\mathbf{F}_z| = 21$ lb.

14. $|\mathbf{F}_x| = 5$ lb.; $|\mathbf{F}_y| = 5$ lb.; $|\mathbf{F}_z| = 13$ lb.

15. $(2, -2, 0)$, $(2, 2, 4)$, $(-1, -1, 1)$.

EXERCISE 3.12 (page 105)

1. (a) 11 units. (b) 11 units. (c) 1 unit. (d) None.
(e) 17 units. (f) 4 units.

3. 39 ft-lb. **4.** $\sqrt{38}$ ft-lb. **5.** 10,824,000 ft-lb.

REVIEW EXERCISE 3 (page 105)

1. (a) Noncollinear. (b) Noncollinear. (c) Noncollinear.

2. (a) Right-angled at Q. (b) Not right-angled. (c) Right-angled at Q.

4. (a) $\sqrt{29}$. (b) $\sqrt{11}$. (c) $\sqrt{21}$. (d) $3\sqrt{10}$. (e) $\sqrt{33}$.

5. (a) $\left(\frac{5}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\right)$. (b) $\left(\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$. (c) $\frac{1}{\sqrt{21}}(4, 1, -2)$.
(d) $\frac{1}{\sqrt{10}}(-3, 1)$. (e) $\frac{1}{\sqrt{33}}(-2, -2, 5)$.

6. (a) $\sqrt{30}$. (b) $\sqrt{34}$. (c) $\sqrt{21}$. (d) $\sqrt{97}$.

7. (a) -5. (b) 7. (c) -17. (d) -22. (e) -22.
(f) 12. (g) 65. (h) 12. (i) 68. (j) -116.

8. (a) $-\frac{1}{42}\sqrt{105}$. (b) $\frac{1}{6}\sqrt{6}$. (c) $-\frac{17}{210}\sqrt{70}$.

9. (a) $\angle P = 72°$, $\angle Q = 52°$, $\angle R = 56°$.
(b) $\angle P = 68°$, $\angle Q = 51°$, $\angle R = 61°$.

10. (a) 37°. (b) 133°. **11.** (a) $\frac{6}{\sqrt{65}}$. (b) $\frac{7}{\sqrt{41}}$.

12. (a) 0. (b) 5. **13.** (a) $\frac{6\sqrt{65}}{13}$ ft-lb. (b) 0.

14. $|\mathbf{F}_x| = \frac{7}{2}\sqrt{6}$; $|\mathbf{F}_y| = \sqrt{6}$; $|\mathbf{F}_z| = \frac{1}{2}\sqrt{6}$. **15.** $-\frac{3}{5}\mathbf{v}_1$, $\frac{6}{5}\mathbf{v}_2$, $\frac{7}{4}\mathbf{v}_3$.

Chapter 4

EXERCISE 4.1 (page 110)

1. (a) $AP:PB = 1:2$. (b) $AB:BP = 1:1$. (c) $PA:AB = 1:1$.
(d) $AB:BP = 4:1$. (e) $PA:AB = 3:2$.
(f) $AB:BP = 1:(\sqrt{2} - 1)$.

2. (a) $\frac{2}{5}$. (b) 2. (c) -1.

3. (a) $AP:PB = 1:1$. (b) $AP:PB = 2:1$. (c) $AP:PB = 4:1$.
(d) $AB:BP = 1:1$. (e) $AB:BP = 2:1$. (f) $PA:AB = 2:1$.

8. (a) $k = \frac{1}{2}$. (b) $k = \frac{2}{3}$. (c) $k = \frac{4}{5}$. (d) $k = 2$.
(e) $k = \frac{3}{2}$. (f) $k = -2$.

EXERCISE 4.2 (page 114)

1. (a) $(x, y) = (1 - k)(1, 1) + k(5, 2)$; $x = 1 + 4k$, $y = 1 + k$.
(b) $(x, y) = (1 - k)(3, 7) + k(-4, 2)$; $x = 3 - 7k$, $y = 7 - 5k$.

(c) $(x, y) = (1 - k)(6, 6) + k(0, 0)$; $x = 6 - 6k$, $y = 6 - 6k$.
(d) $(x, y) = (1 - k)(-1, 0) + k(0, 7)$; $x = -1 + k$, $y = 7k$.
(e) $(x, y) = (1 - k)(3, -3) + k(0, 5)$; $x = 3 - 3k$, $y = -3 + 8k$.
(f) $(x, y) = (1 - k)(-2, 3) + k(-4, 7)$; $x = -2 - 2k$, $y = 3 + 4k$.
(g) $(x, y) = (1 - k)(-2, 4) + k(-5, -6)$; $x = -2 - 3k$, $y = 4 - 10k$.
(h) $(x, y) = (1 - k)(-3, -1) + k(-7, -2)$; $x = -3 - 4k$, $y = -1 - k$.
(i) $(x, y) = (1 - k)(4, \frac{1}{3}) + k(-\frac{1}{2}, 0)$; $x = 4 - \frac{9}{2}k$, $y = \frac{1}{3} - \frac{1}{3}k$.
(j) $(x, y) = (1 - k)(5, \sqrt{2}) + k(\sqrt{3}, 6)$; $x = 5 + k(\sqrt{3} - 5)$, $y = \sqrt{2} + k(6 - \sqrt{2})$.

2. (a) $(3, \frac{3}{2})$. (b) $(-\frac{1}{2}, \frac{9}{2})$. (c) $(3, 3)$. (d) $(-\frac{1}{2}, \frac{7}{2})$.
(e) $(\frac{3}{2}, 1)$. (f) $(-3, 5)$. (g) $(-\frac{7}{2}, -1)$. (h) $(-5, -\frac{3}{2})$.
(i) $(\frac{7}{4}, \frac{1}{6})$. (j) $\left(\frac{5}{2} + \frac{\sqrt{3}}{2}, 3 + \frac{\sqrt{2}}{2}\right)$.

3. (a) $(-\frac{1}{3}, 3)$. (b) $(\frac{1}{4}, \frac{25}{3})$. (c) $(-19, -1)$. (d) $(23, 8)$.

4. (a) $AP : PB = 2 : 1$. (b) $PA : AB = 1 : 1$. (c) $PA : AB = 1 : 3$.
(d) $AB : BP = 1 : 4$.

5. (a) Line, $11x + 7y = 9$. (b) Point, $(4, -5)$.
(c) Point, $(-3, 6)$. (d) Point $(\frac{1}{2}, \frac{1}{2})$.
(e) Point, $(-10, 17)$. (f) Three points, $(4, -5)$, $(-3, 6)$, $(-10, 17)$.
(g) Three points, $(11, -16)$, $(4, -5)$, $(-10, 17)$. (h) Line segment.
(i) Ray. (j) Ray. (k) Two rays.

6. $(5, 4)$, $(2, 11)$ are on the line; $(14, -17)$ is on the line; $(-\frac{1}{2}, \frac{27}{2})$ is not.

8. (a) $(x, y) = (2, 3) + k(3, 5)$; $x = 2 + 3k$, $y = 3 + 5k$.
(b) $(x, y) = (2, 3) + k(-2, 1)$; $x = 2 - 2k$, $y = 3 + k$.
(c) $(x, y) = (2, 3) + k(0, -5)$; $x = 2$, $y = 3 - 5k$.
(d) $(x, y) = (2, 3) + k(-4, 0)$; $x = 2 - 4k$, $y = 3$.
(e) $(x, y) = (2, 3) + k(2, 6)$; $x = 2 + 2k$, $y = 3 + 6k$.
(f) $(x, y) = (2, 3) + k(-7, -4)$; $x = 2 - 7k$, $y = 3 - 4k$.

9. (a) $(3, 4)$, $(5, 1)$; $(2, -3)$. (b) $(-1, 3)$, $(-3, 2)$; $(-2, -1)$.
(c) $(0, 1)$, $(4, -1)$; $(4, -2)$. (d) $(3, -2)$, $(3, 3)$; $(0, -5)$.
(e) $(0, 0)$, $(1, -2)$; $(1, -2)$. (f) $(4, -1)$, $(3, -1)$; $(-1, 0)$.

12. $x = kx_1$, $y = ky_1$. **13.** (a) $x = x_1 + k$, $y = y_1$. (b) $x = x_1$, $y = y_1 + k$.

EXERCISE 4.3 (page 120)

1. (a) $x + y = 10$. (b) $x + y = -1$. (c) $x - 4y = 34$.
(d) $-x + 2y = 2$. (e) $3x - y = 30$. (f) $2x + 3y = 13$.
(g) $x = 10$. (h) $y = 9$. (i) $3x + 2y = 13$.
(j) $2x + 5y = 3$. (k) $3x + 7y = 0$. (l) $2x - 5y = -25$.

2. (a) $3x + 5y = 34$. (b) $3x - 7y = -58$. (c) $2x + 7y = -53$.
(d) $y = 4$. (e) Not determined.

3. (a) $3x + 4y = \pm 25$. (b) $-2x + 7y = \pm 3\sqrt{53}$.
(c) $2x - 3y = \pm 10\sqrt{13}$.

4. (a) $(\frac{3}{2}\frac{9}{5}, \frac{5}{2}\frac{2}{5})$. (b) $(\frac{7}{3}, 0)$. (c) $\left(\frac{18}{29}, \frac{-45}{29}\right)$. (d) $\left(\frac{-9}{58}, \frac{21}{58}\right)$.
(e) $\left(0, \frac{-5}{2}\right)$. (f) $\left(\frac{-3}{13}, \frac{9}{26}\right)$. (g) $(-3, 2)$. (h) $(\frac{3}{2}\frac{6}{}, \frac{1}{2}\frac{5}{6})$.

5. (a) $\frac{1}{5}\frac{3}{}$. (b) $\frac{7}{3}$. (c) $\frac{9\sqrt{29}}{29}$. (d) $\frac{3\sqrt{58}}{58}$. (e) $\frac{5}{2}$. (f) $\frac{3\sqrt{13}}{26}$. (g) $\sqrt{13}$. (h) $\frac{3\sqrt{26}}{26}$.

6. (a) $\frac{8}{\sqrt{5}}$. (b) 0. (c) $\frac{21}{\sqrt{34}}$. (d) $\frac{28}{\sqrt{53}}$.

EXERCISE 4.4 (page 124)

1. (a) $(4, 6), \frac{x-3}{2} = \frac{y-3}{3}$. (b) $(10, -5), \frac{x+2}{2} = \frac{y-5}{-1}$.
(c) $(1, 3), \frac{x+3}{1} = \frac{y+2}{3}$. (d) $(1, 1), \frac{x+1}{1} = \frac{y+2}{1}$.
(e) $(7, -3), \frac{x-1}{7} = \frac{y-1}{-3}$. (f) $(9, 2), \frac{x}{9} = \frac{y-2}{2}$.
(g) $(3, 4), \frac{x-3}{3} = \frac{y}{4}$. (h) $(3, -2), \frac{x-4}{3} = \frac{y-6}{-2}$.
(i) $(-1, 0)$, no symmetric equation. (j) $(0, -3)$, no symmetric equation.
(k) $(4, 3), \frac{x-5}{4} = \frac{y-2}{3}$.

2. (a) $x + 5y = -24$. (b) $7x - 3y = 22$. (c) $3x - 4y = 23$.
(d) $y + 5 = 0$.

3. $5x + 3y + 1 = 0$. **4.** (a) Parallel to x-axis. (b) Parallel to y-axis.

EXERCISE 4.5 (page 128)

1. (a) 56°, 34°. (b) 153°, −63°. (c) 72°, 18°. (d) 45°, 45°.
(e) 157°, −67°. (f) 13°, 77°. (g) 53°, 37°. (h) 146°, −56°.
(i) 180°, −90°. (j) 90°, 0°. (k) 37°, 53°.

2. (a) $x - y = -1$. (b) $2\sqrt{3} + 3 = \sqrt{3}x + y$. (c) $.259x + .966y = 3.416$.

3. (a) 41°, 49°. (b) 109°, −19°. (c) 127°, −37°. (d) 67°, 23°.

4. $\sqrt{3}x + y = \pm 10$.

EXERCISE 4.6 (page 131)

1. (a) $(x, y, z) = (1 - r)(1, 1, 1) + r(5, 5, 5)$; $x = k$, $y = k$, $z = k$; $x = y = z$.
(b) $\frac{x-1}{4} = \frac{y}{3} = \frac{z-4}{5}$. (c) $x = 4 - 3r$, $y = 2$, $z = 1 + 3r$.
(d) $\frac{x+1}{8} = \frac{y-3}{4} = \frac{z-5}{-8}$. (e) $\frac{x+4}{3} = \frac{y-1}{4} = \frac{z-7}{-5}$.
(f) $x = 4 + 2r$, $y = -3 + 10r$, $z = -3$.
(g) $x = 5 - 7r$, $y = -1$, $z = 3$. (h) $\frac{x+6}{5} = \frac{y-2}{-3} = \frac{z-1}{-8}$.
(i) $x = -3$, $y = 1 + 5r$, $z = 6$. (j) $x = 0$, $y = r$, $z = 1 - r$.

2. (a) $(3, 3, 3)$. (b) $(3, \frac{3}{2}, \frac{13}{2})$. (c) $(\frac{5}{2}, 2, \frac{5}{2})$. (d) $(3, 5, 1)$.
(e) $(-\frac{5}{2}, 3, \frac{9}{2})$. (f) $(5, 2, -3)$. (g) $(\frac{3}{2}, -1, 3)$. (h) $(-\frac{7}{2}, \frac{1}{2}, -3)$.
(i) $(-3, \frac{7}{2}, 6)$. (j) $(0, \frac{1}{2}, \frac{1}{2})$.

3. (a) $(2, \frac{1}{2}, 0)$. (b) $(\frac{13}{7}, \frac{5}{7}, \frac{2}{7})$. (c) No such point. (d) $(-2, \frac{13}{2}, 8)$.

4. (a) $AP:PB = 1:1$. (b) $AP:PB = 4:1$. (c) $PA:AB = 1:1$.
(d) $AP:PB = 4:-3$. (e) $AB:BP = 1:1$. (f) $AP:PB = -4:5$.

6. (a) Parallel to y-axis through $(3, 0, -1)$.
(b) Parallel to x-axis through $(0, 4, -2)$.
(c) In the plane $z = 2$ through $(0, \frac{17}{2}, 2)$ and $(\frac{17}{3}, 0, 2)$.

7. (a) $x = x_1 + t, y = y_1, z = z_1$. (b) $x = x_1, y = y_1 + t, z = z_1$.
(c) $x = x_1, y = y_1, z = z_1 + t$.

8. $x = x_1 + ks_1, y = y_1 + ks_2, z = z_1 + ks_3$.

9. (a) $x = -2 - 3k, y = 5 + k, z = 3 - 4k$.
(b) $x = -2 + 4k, y = 5, z = 3 - 7k$.
(c) $x = -2, y = 5 - 2k, z = 3 + k$.

EXERCISE 4.7 (page 136)

1. (a) 55°, 55°, 55°. (b) 16°, 79°, 79°. (c) 45°, 90°, 45°.
(d) 135°, 90°, 45°. (e) 52°, 99°, 40°. (f) 135°, 124°, 65°.
(g) 29°, 103°, 64°. (h) 126°, 79°, 38°. (i) 146°, 90°, 33°.
(j) 34°, 124°, 90°.

2. (a) Parallel to z-axis. (b) Parallel to y-axis. (c) Parallel to x-axis.

3. (a) $\dfrac{x-1}{4} = \dfrac{y-1}{3} = \dfrac{z-5}{-1}$. (b) $\dfrac{x+3}{1} = \dfrac{y-1}{-2} = \dfrac{z-3}{4}$.
(c) $x = 4 + 3t, y = 2, z = 4 - 2t$. (d) $x = -1, y = 5 + 3t, z = 2$.
(e) $\dfrac{x-4}{1} = \dfrac{y-1}{-2} = \dfrac{z-7}{2}$. (f) $x = 2 + k, y = -1 + 2k, z = 8$.
(g) $x = 4 + k, y = 1 + k, z = 0$. (h) $\dfrac{x+2}{1} = \dfrac{y-1}{-1} = \dfrac{z-2}{\sqrt{2}}$.

REVIEW EXERCISE 4 (page 137)

1. (a) $AP:PB = 3:1$. (b) $AP:PB = -4:7$.
(c) Segment PB, where $AP:PB = 1:1$.
(d) Ray AP, where $PA:AB = 1:1$.

2. (a) $k = \frac{1}{5}$. (b) $k = -\frac{1}{2}$.

3. (a) P is A or B. (b) P is on ray BC with B between A and C.
(c) Line segment CB with $CA:AB = 1:1$.

4. (a) $\mathbf{r} = (1 - k)(4, 1) + k(3, 4); x = 4 - k, y = 1 + 3k$.
(b) $\mathbf{r} = (1 - k)(-2, 1) + k(3, -2); x = -2 + 5k, y = 1 - 3k$.
(c) $\mathbf{r} = (1 - k)(5, 0) + k(0, 2); x = 5 - 5k, y = 2k$.
(d) $\mathbf{r} = (1 - k)(-3, -4) + k(-2, 0); x = -3 + k, y = -4 + 4k$.

5. (a) $A(-3, 5), B(4, 0)$. (b) Ray AB. (c) Line AB.

6. (a) $\mathbf{r} = (2, -1) + k(3, -1)$; $x = 2 + 3k$, $y = -1 - k$.
(b) $\mathbf{r} = (5, 0) + k(-2, -3)$; $x = 5 - 2k$, $y = 0 - 3k$.

7. (a) $2x + 3y = 5$. (b) $x = -2$. (c) $5x - 2y = 0$. (d) $x + y = 6$.

8. (a) $5x + 2y = 29$. (b) $3x - y = -10$. (c) $4x + 3y = -25$.

9. (a) $(\frac{9}{17}, \frac{36}{17})$. (b) $(\frac{7}{2}, 0)$. (c) $(\frac{-12}{13}, \frac{8}{13})$. (d) $(\frac{3}{5}, -\frac{9}{5})$.

10. (a) $\frac{9}{17}\sqrt{17}$. (b) $\frac{7}{2}$. (c) $\frac{4}{13}\sqrt{13}$. (d) $\frac{3}{5}\sqrt{10}$.

11. (a) $\frac{10}{17}\sqrt{17}$. (b) $\frac{9}{2}$. (c) $\frac{9}{13}\sqrt{13}$. (d) $\frac{11}{5}\sqrt{10}$.

12. (a) $(7, -7)$; $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; $(135°, -45°)$; $\frac{x+3}{1} = \frac{y-4}{-1}$.

(b) $(2, 3)$; $\left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$; $(56°, 34°)$; $\frac{x-4}{2} = \frac{y}{3}$.

(c) $(2, -1)$; $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$; $(153°, -63°)$; $\frac{x-5}{2} = \frac{y+3}{-1}$.

(d) $(4, -3)$; $(-\frac{4}{5}, \frac{3}{5})$; $(143°, -53°)$; $\frac{x-1}{4} = \frac{y-1}{-3}$.

13. (a) $(x, y, z) = (1 - k)(3, -2, 3) + k(4, 0, -2)$; $x = 3 + k$, $y = -2 + 2k$, $z = 3 - 5k$; $\frac{x-3}{1} = \frac{y+2}{2} = \frac{z-3}{-5}$.

(b) $(x, y, z) = (1 - k)(5, 2, -1) + k(5, 2, 6)$; $x = 5$, $y = 2$, $z = -1 + 7k$.

(c) $(x, y, z) = (1 - k)(-1, 3, -2) + k(5, -3, -3)$; $x = -1 + 6k$, $y = 3 - 6k$, $z = -2 - k$; $\frac{x+1}{6} = \frac{y-3}{-6} = \frac{z+2}{-1}$.

(d) $(x, y, z) = (1 - k)(2, 1, 7) + k(-6, 1, -4)$; $x = 2 - 8k$, $y = 1$, $z = 7 - 11k$.

14. (a) $(\frac{7}{2}, -1, \frac{1}{2})$. (b) $(5, 2, \frac{5}{2})$. (c) $(2, 0, -\frac{5}{2})$. (d) $(-2, 1, \frac{3}{2})$.

15. (a) $(-1, -2, 5)$; $\left(\frac{-1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$; $(101°, 111°, 24°)$.

(b) $(0, 0, 7)$; $(0, 0, 1)$; $(90°, 90°, 0°)$.

(c) $(6, -6, -1)$; $\left(-\frac{6}{\sqrt{73}}, \frac{6}{\sqrt{73}}, \frac{1}{\sqrt{73}}\right)$; $(135°, 45°, 83°)$.

(d) $(8, 0, 11)$; $\left(\frac{8}{\sqrt{185}}, 0, \frac{11}{\sqrt{185}}\right)$; $(54°, 90°, 36°)$.

16. (a) $(3, -7, 3)$; $\left(\frac{3}{\sqrt{67}}, \frac{-7}{\sqrt{67}}, \frac{3}{\sqrt{67}}\right)$; $(68°, 149°, 68°)$.

(b) $(5, 1, 0)$; $\left(\frac{5}{\sqrt{26}}, \frac{1}{\sqrt{26}}, 0\right)$; $(11°, 79°, 90°)$.

17. (a) $\frac{x-4}{5} = \frac{y-1}{1} = \frac{z+2}{-4}$. (b) $x = -2 + \frac{3}{5}k$, $y = 3$, $z = -1 + \frac{4}{5}k$.

(c) $x = 5 + k$, $y = -1 + \sqrt{2}k$, $z = 2 - k$.

Chapter 5

EXERCISE 5.1 (page 141)

2. (a) $m = \frac{1}{2}$. (b) $k = -4$. (c) $k + m = 3$. (d) $k - m = -3$.

EXERCISE 5.2 (page 143)

1. (a) $(x, y, z) = (1 - k - m)(0, 0, 0) + k(1, 4, 9) + m(2, 2, 5)$;
$x = k + 2m$, $y = 4k + 2m$, $z = 9k + 5m$.
(b) $(x, y, z) = (1 - k - m)(0, 0, 0) + k(-1, -3, -1) + m(2, 5, -4)$;
$x = -k + 2m$, $y = -3k + 5m$, $z = -k - 4m$.
(c) $(x, y, z) = (1 - k - m)(1, 0, 0) + k(0, 1, 0) + m(0, 0, 1)$;
$x = 1 - k - m$, $y = k$, $z = m$.
(d) $(x, y, z) = (1 - k - m)(0, 0, 3) + k(0, -2, 0) + m(5, 0, 0)$;
$x = 5m$, $y = -2k$, $z = 3 - 3k - 3m$.
(e) $(x, y, z) = (1 - k - m)(5, 0, 0) + k(3, 2, -1) + m(4, 4, 0)$;
$x = 5 - 2k - m$, $y = 2k + 4m$, $z = -k$.
(f) $(x, y, z) = (1 - k - m)(-1, 3, -1) + k(0, 2, 0) + m(3, -1, 2)$;
$x = -1 + k + 4m$, $y = 3 - k - 4m$, $z = -1 + k + 3m$.
(g) $(x, y, z) = (1 - k - m)(2, 4, 3) + k(2, -1, 5) + m(2, 0, 6)$;
$x = 2$, $y = 4 - 5k - 4m$, $z = 3 + 2k + 3m$.

3. (a) $x = x_1$, $y = k$, $z = m$. (b) $x = k$, $y = y_1$, $z = m$.
(c) $x = k$, $y = m$, $z = z_1$.

EXERCISE 5.3 (page 148)

1. (a) $5x + 5y + z = 15$. (b) $y + 2z = 7$.
(c) $5x + 2y = -13$. (d) $x = 3$. (e) $4x + 2y - z = 3$.
(f) $-3x - 2y + 2z = -11$. (g) $3x + 3y - 4z = 0$.
(h) $x = 0$. (i) $3x + 5y - 2z = 6$. (j) $y = 0$.

2. (a) $y = \pm 4$. (b) $x = \pm 2$. (c) $5x - 3z = \pm 3\sqrt{34}$.
(d) $5y - 2z = \pm \frac{1}{2}\sqrt{29}$. (e) $3x - 3y + z = \pm\sqrt{57}$.
(f) $2x - 3y - 5z = \pm\pi\sqrt{38}$.

3. (a) $2x + 2y + 3z = 17$. (b) $-4x + 3y = 25$. (c) $4x - y - 3z = 26$.

4. (a) $(0, 0, 0)$. (b) $(\frac{5}{26}, \frac{15}{26}, \frac{10}{13})$. (c) $(\frac{-10}{11}, \frac{30}{11}, \frac{10}{11})$.
(d) $(\frac{80}{27}, \frac{-16}{27}, \frac{-16}{27})$. (e) $(\frac{-24}{7}, \frac{36}{7}, \frac{-12}{7})$. (f) $(\frac{-24}{13}, \frac{8}{13}, \frac{-32}{13})$.

5. (a) 0. (b) $\frac{5}{26}\sqrt{26}$. (c) $\frac{10}{11}\sqrt{11}$. (d) $\frac{16}{9}\sqrt{3}$. (e) $\frac{12}{7}\sqrt{14}$. (f) $\frac{8}{13}\sqrt{26}$.

7. (a) $\frac{6}{7}\sqrt{14}$. (b) $\frac{1}{2}\sqrt{14}$. (c) $\frac{17}{19}\sqrt{38}$. (d) $\frac{1}{6}\sqrt{18}$. (e) $\frac{29}{14}\sqrt{14}$. (f) $\frac{37}{29}\sqrt{29}$.

REVIEW EXERCISE 5 (page 159)

3. (a) $x = 3k + 4m$, $y = 2k - m$, $z = -k + m$.
(b) $x = 1 + 2k - 3m$, $y = 2 - 3k - m$, $z = -1 + 5k + 6m$.
(c) $x = -3 + 9k$, $y = 1 - 2k + 4m$, $z = 4 - 5k - 4m$.
(d) $x = 5 - 8k - m$, $y = -2 + 2k + 2m$, $z = -m$.

5. (a) $x + y + 5z = 14$. (b) $4x - y = 6$.
(c) $-x + 3y - z = -10$. (d) $2x + 5z = 26$.

6. (a) $5x + y + 2z = \pm 3\sqrt{30}$. (b) $2x - y - z = \pm 3\sqrt{2}$.

7. (a) $(\frac{3}{7}, \frac{-9}{7}, \frac{6}{7})$. (b) $\frac{3}{7}\sqrt{14}$.

8. (a) $\frac{6}{13}\sqrt{26}$. (b) $\frac{9}{41}\sqrt{41}$. (c) $\frac{3}{5}\sqrt{10}$. (d) 1.

Chapter 6

EXERCISE 6.1 (page 163)

1. $\{(-7, 15)\}$. **2.** $\{(1, 2)\}$. **3.** $\{(3, 5)\}$. **4.** $\{(1, 2)\}$.
5. $\{(\frac{28}{19}, \frac{30}{19})\}$. **6.** $\{(1, 3)\}$. **7.** $\{(\frac{1}{2}, 1)\}$. **8.** $\{(\frac{1}{2}, -\frac{11}{4})\}$.
9. $\{(6, -5)\}$. **10.** $\{(-3, 5, 7)\}$. **11.** $\{(2, -1, 1)\}$. **12.** $\{(2, 3, 4)\}$.
13. $\{(-\frac{10}{17}, \frac{39}{17}, \frac{130}{17})\}$. **14.** $\{(-1, -1, -1)\}$.
15. $\{(2, -2, \frac{1}{2})\}$. **16.** $\{(\frac{1}{6}, \frac{-1}{2})\}$. **17.** $\{(\frac{1}{3}, \frac{1}{2})\}$. **18.** $\{(0, 0, 0)\}$.
19. $\{(\frac{11}{2}, \frac{11}{30})\}$. **20.** $\{(\frac{11}{140}, \frac{-11}{57})\}$. **21.** $\{(0, 0, 0)\}$.

EXERCISE 6.2 (page 166)

1. $\{(\frac{29}{3}, 11)\}$. **2.** Inconsistent. **3.** Inconsistent. **4.** Inconsistent.
5. $\{(\frac{18}{19}, \frac{28}{19}, \frac{31}{19})\}$. **6.** Inconsistent. **7.** Inconsistent. **8.** $\{(2, -1)\}$.
9. Inconsistent. **10.** Inconsistent. **11.** Inconsistent. **12.** $\{(\frac{4}{3}, \frac{4}{15}, \frac{-17}{15}\})$.
13. Inconsistent. **14.** $\{(\frac{18}{7}, \frac{5}{7})\}$. **15.** Inconsistent.

EXERCISE 6.3 (page 170)

1. Dependent; $\{(4 + 3k, k) \mid k \in Re\}$.

2. Dependent; $\left\{\left(-\dfrac{2k + 3}{3}, k\right) \;\middle|\; k \in Re\right\}$.

3. Dependent; $\{(3 - 2k + l, k, l) \mid k, l \in Re\}$.

4. Dependent; $\left\{\left(\dfrac{k - 3l - 1}{2}, k, l\right) \;\middle|\; k, l \in Re\right\}$.

5. Dependent; $\{(5 - k + l - m, k, l, m) \mid k, l, m \in Re\}$.

6. Dependent; $\{(3k - 2l + m - 2, k, l, m) \mid k, l, m \in Re\}$.

7. Independent; $\{(\frac{7}{5}, \frac{4}{5})\}$. **8.** Inconsistent.

9. Dependent; $\{(3 - 2k, k) \mid k \in Re\}$. **10.** Inconsistent.

11. Dependent; $\left\{\left(-5k - 13, -\dfrac{7k + 19}{2}, k\right) \;\middle|\; k \in Re\right\}$.

12. Independent; $\{(2, 1, -3)\}$.

13. Dependent; $\left\{\left(\dfrac{9 - 5k}{3}, -\dfrac{k}{3}, k\right) \;\middle|\; k \in Re\right\}$.

14. Dependent; $\left\{\left(\dfrac{31 - 4k}{7}, \dfrac{5k - 9}{7}, k\right) \;\middle|\; k \in Re\right\}$.

15. Dependent; $\left\{\left(\dfrac{8k - 8}{7}, \dfrac{26k - 19}{7}, k\right) \;\middle|\; k \in Re\right\}$.

16. Dependent; $\left\{\left(\dfrac{5k - 8l + 24}{7}, \dfrac{k - 3l + 9}{7}, k, l\right) \;\middle|\; k, l \in Re\right\}$.

17. Dependent; $\left\{\left(\frac{-17k+13l-20}{5}, \frac{-13k+17l-25}{5}, k, l\right) \mid k, l \in Re\right\}$.

18. Dependent; $\left\{\left(\frac{-7+8k-7l}{10}, \frac{9-k+4l}{5}, k, l\right) \mid k, l \in Re\right\}$.

19. Dependent; $\left\{\left(\frac{11-2k-7l+13m}{5}, \frac{3k+3l-2m+1}{5}, k, l, m\right) \mid k, l, m \in Re\right\}$.

20. $\left\{\left(\frac{4-5k}{2}, k\right) \mid k \in Re\right\}$.

21. $\left\{\left(\frac{8-k}{3}, \frac{1+7k}{3}, k\right) \mid k \in Re\right\}$.

22. $\left\{\left(\frac{6-2k}{5}, \frac{2+16k}{5}, k\right) \mid k \in Re\right\}$.

EXERCISE 6.4 (page 173)

1. Dependent; $\{(5k, k) \mid k \in Re\}$.

2. Dependent; $\left\{\left(\frac{2k-l}{3}, k, l\right) \mid k, l \in Re\right\}$.

3. Dependent; $\{(2k+l-3m, k, l, m) \mid k, l, m \in Re\}$.

4. Independent; $\{(0, 0)\}$.

5. Dependent; $\{(-3k, k) \mid k \in Re\}$.

6. Independent; $\{(0, 0)\}$.

7. Dependent; $\{(2k, k, 0) \mid k \in Re\}$.

8. Dependent; $\{(2k-l, k, l) \mid k, l \in Re\}$.

9. Dependent; $\{(-k, -k, k) \mid k \in Re\}$.

10. Independent; $\{(0, 0, 0)\}$.

11. Dependent; $\left\{\left(-\frac{k}{7}, \frac{4k}{7}, k\right) \mid k \in Re\right\}$.

12. Dependent; $\{(-2k+l, k, l) \mid k, l \in Re\}$.

13. Dependent; $\{(l-k, k-2l, k, l) \mid k, l \in Re\}$.

14. Dependent; $\{(4k-3l, 11k-7l, k, l) \mid k, l \in Re\}$.

15. Dependent; $\left\{\left(\frac{k+l-2m}{2}, k, l, m\right) \mid k, l, m \in Re\right\}$.

16. Dependent; $\left\{\left(\frac{-2k+l}{3}, k, l, 0\right) \mid k, l \in Re\right\}$.

17. Dependent; $\{(-k, -k, k, 0) \mid k \in Re\}$.

18. Dependent; $\left\{\left(\frac{-9k}{28}, \frac{-43k}{28}, \frac{-87k}{28}, k\right) \mid k \in Re\right\}$.

EXERCISE 6.5 (page 176)

1. (1) $[1 \quad -3 \quad 4]$. (2) $[3 \quad 2 \quad -3]$. (3) $[1 \quad 2 \quad -1 \quad 3]$.
(4) $[2 \quad -1 \quad 3 \quad -1]$. (5) $[1 \quad 1 \quad -1 \quad 1 \quad 5]$. (6) $[1 \quad -3 \quad 2 \quad -1 \quad -2]$.
(7) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \end{bmatrix}$. (8) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 6 \end{bmatrix}$. (9) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$.

(10) $\begin{bmatrix} 1 & -2 & -2 & 6 \\ 3 & -4 & 1 & -1 \\ 5 & -8 & -3 & 12 \end{bmatrix}$. (11) $\begin{bmatrix} 1 & -2 & -2 & 6 \\ 3 & -4 & 1 & -1 \\ 5 & -8 & -3 & 11 \end{bmatrix}$. (12) $\begin{bmatrix} 1 & -2 & -2 & 6 \\ 3 & -4 & 1 & -1 \\ 2 & -5 & 3 & -10 \end{bmatrix}$.

(13) $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -1 & 3 & 6 \end{bmatrix}$. (14) $\begin{bmatrix} 2 & 3 & -1 & 5 \\ 1 & 5 & -3 & -2 \end{bmatrix}$. (15) $\begin{bmatrix} 3 & -2 & 4 & 2 \\ 5 & -1 & -2 & -3 \end{bmatrix}$.

(16) $\begin{bmatrix} 1 & 2 & -1 & 2 & 6 \\ 3 & -1 & -2 & 3 & 9 \end{bmatrix}$. (17) $\begin{bmatrix} 2 & -3 & -1 & 5 & 7 \\ 3 & -2 & 5 & -1 & -2 \end{bmatrix}$.

(18) $\begin{bmatrix} 2 & 3 & -1 & -1 & 4 \\ 4 & 1 & -3 & 2 & -1 \end{bmatrix}$. (19) $\begin{bmatrix} 1 & -1 & 1 & 2 & -3 & 2 \\ 2 & 3 & -1 & 1 & -4 & 5 \end{bmatrix}$

(20) $[2 \quad 5 \quad 4]$. (21) $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 5 \end{bmatrix}$. (22) $\begin{bmatrix} 3 & 1 & -2 & 4 \\ 2 & -1 & 4 & 2 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 3 & -2 & 7 \\ 5 & 2 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 5 & 2 & 4 \\ 3 & -2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 2 & 4 \\ 23 & 6 & 23 \end{bmatrix} \rightarrow \begin{bmatrix} 30 & 12 & 24 \\ 23 & 6 & 23 \end{bmatrix}$.

3. $\begin{bmatrix} 5 & 0 \\ 51 & -3 \end{bmatrix}$. **4.** $\begin{bmatrix} 1 & 1 \\ 18 & 0 \end{bmatrix}$. **5.** $\begin{bmatrix} 0 & 1 \\ 3 & 9 \end{bmatrix}$. **6.** $\begin{bmatrix} 0 & 1 & 3 \\ 12 & 15 & 24 \end{bmatrix}$.

7. $\begin{bmatrix} 4 & 1 & 6 \\ 42 & 12 & 48 \end{bmatrix}$. **8.** $\begin{bmatrix} 1 & 1 & 1 \\ 12 & 12 & 12 \end{bmatrix}$. **9.** $\begin{bmatrix} 2 & 1 & 5 & 2 \\ 27 & 6 & 60 & 18 \end{bmatrix}$.

10. $\begin{bmatrix} 0 & 4 & 1 & 5 \\ -9 & 39 & 12 & 51 \end{bmatrix}$. **11.** $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 15 & 9 & 18 & 15 \end{bmatrix}$. **12.** $\begin{bmatrix} 10 & 2 & 4 \\ 0 & 1 & 5 \\ 33 & 5 & 16 \end{bmatrix}$.

13. $\begin{bmatrix} 2 & 2 & 4 \\ 5 & 3 & 1 \\ 8 & 5 & 18 \end{bmatrix}$. **14.** $\begin{bmatrix} 8 & 2 & 2 & -6 \\ 0 & 1 & 5 & 2 \\ 25 & 9 & 4 & -17 \end{bmatrix}$. **15.** $\begin{bmatrix} 1 & -1 & 4 \\ \frac{17}{2} & \frac{-9}{2} & 14 \\ 1 & 4 & -1 \end{bmatrix}$.

16. $\begin{bmatrix} 0 & -1 & -2 \\ 4 & -\frac{9}{2} & -5 \\ 3 & 1 & 5 \end{bmatrix}$. **17.** $\begin{bmatrix} 0 & -4 & 2 & -1 \\ 4 & 10 & -7 & -\frac{9}{2} \\ 2 & 1 & 3 & 5 \end{bmatrix}$. **18.** $\begin{bmatrix} -2 & 7 & 3 \\ 11 & -\frac{9}{2} & \frac{3}{2} \\ -2 & 1 & -1 \end{bmatrix}$.

19. $\begin{bmatrix} 4 & 2 & 0 \\ 10 & -11 & 12 \\ 3 & 1 & -4 \end{bmatrix}$. **20.** $\begin{bmatrix} -3 & 3 & -3 & -1 \\ \frac{29}{2} & -\frac{13}{2} & \frac{37}{2} & -\frac{9}{2} \\ 3 & -1 & 3 & 5 \end{bmatrix}$.

EXERCISE 6.6 (page 181)

1. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. **2.** $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$. **3.** $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -\frac{7}{2} \end{bmatrix}$.

5. $\begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & \frac{23}{11} \end{bmatrix}$. **6.** $\begin{bmatrix} 1 & \frac{-1}{2} & 3 \\ 0 & 1 & \frac{-22}{9} \end{bmatrix}$. **7.** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. **9.** $\begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. **10.** $\begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -\frac{11}{4} \\ 0 & 0 & 1 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & 1 & \frac{-11}{19} \\ 0 & 0 & 0 \end{bmatrix}$. **12.** $\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & \frac{23}{19} \\ 0 & 0 & 0 \end{bmatrix}$. **13.** $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & \frac{1}{12} & \frac{1}{3} \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

14. $\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{7}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{5}{13} \end{bmatrix}$. **15.** $\begin{bmatrix} 1 & 7 & 1 & -3 \\ 0 & 1 & \frac{1}{6} & -\frac{4}{9} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. **16.** $\begin{bmatrix} 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{5}{17} \end{bmatrix}$.

17. $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & -\frac{11}{9} \\ 0 & 0 & 1 \end{bmatrix}$. **18.** $\begin{bmatrix} 1 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{7}{4} & \frac{5}{4} \\ 0 & 0 & 1 & \frac{13}{33} \end{bmatrix}$. **19.** $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \end{bmatrix}$.

20. $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -\frac{4}{9} \\ 0 & 0 & 1 \end{bmatrix}$. **21.** $\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & -\frac{8}{3} \\ 0 & 0 & 1 & -\frac{82}{69} \end{bmatrix}$.

EXERCISE 6.7 (page 186)

1. $x = 5$; $\{5\}$; independent.

2. $x + 3y = -2$; $\{(-3k - 2, k) \mid k \in Re\}$; dependent.

3. $x + 2y = -1$, $y = 6$; $\{(-13, 6)\}$; independent.

4. $x = 4$, $y = -3$; $\{(4, -3)\}$; independent.

5. $x + 3y = -1$, $y = 0$; $\{(-1, 0)\}$; independent.

6. $x = 5$, $0y = 4$; ϕ; inconsistent.

7. $x + 3y = -2$, $0x + 0y = 0$; $\{(-3k - 2, k) \mid k \in Re\}$) dependent.

8. $x + 3y = 0$; $y = 0$; $\{(0, 0)\}$; independent.

9. $x + 5y = 0$, $0x + 0y = 0$; $\{(-5k, k) \mid k \in Re\}$; dependent.

10. $x + 2y - 3z = 1$, $z = 2$; $\{(7 - 2k, k, 2) \mid k \in Re\}$; dependent.

11. $x - y + 5z = 2$, $y + z = 4$; $\{(6 - 6k, 4 - k, k) \mid k \in Re\}$; dependent.

12. $x + 3y + 3z = 2$, $0x + 0y + 0z = 4$; ϕ; inconsistent.

13. $x + y - 4z = 3$, $0x + 0y + 0z = 0$; $\{(3 - k + 4l, k, l) \mid k, l \in Re\}$; dependent.

14. $x + 2y - z = 5$, $y + 3z = 1$, $z = 2$; $\{(17, -5, 2)\}$; independent.

15. $x + 3y + 3z = 2$, $y + 4z = -1$; $0x + 0y + 0z = 1$; ϕ; inconsistent.

16. $x + 2y - z = 5$, $z = 4$, $0x + 0y + 0z = 1$; ϕ; inconsistent.

17. $x + 2y - z = 5$, $z = 4$, $0x + 0y + 0z = 0$; $\{9 - 2k, k, 4) \mid k \in Re\}$; dependent.

18. $x + 3y + z = 4$, $y + 5z = 2$, $0x + 0y + 0z = 0$;
$\{(14k - 2, 2 - 5k, k) \mid k \in Re\}$; dependent.

19. $\{(\frac{18}{11}, \frac{5}{11})\}$, independent. **20.** ϕ, inconsistent.

21. $\left\{\left(\frac{4 + k}{3}, k\right) \middle| k \in Re\right\}$, dependent.

22. $\left\{\left(\frac{1-k}{7}, \frac{24+4k}{7}, k\right) \mid k \in Re\right\}$, dependent.

23. $\left\{\left(\frac{5-k}{2}, \frac{3k-5}{2}, k\right) \mid k \in Re\right\}$, dependent

24. $\{(\frac{8}{7}, \frac{-19}{7})\}$, independent. **25.** $\{(1, \frac{2}{5}, \frac{-12}{5})\}$, independent.

26. ϕ, inconsistent. **27.** $\left\{\left(\frac{8-5k}{11}, \frac{13+7k}{11}, k\right) \mid k \in Re\right\}$, dependent.

EXERCISE 6.8 (page 190)

2. $\{(x, y) \mid x = -3k - 2, y = k, k \in Re\}$, line.

7. $\{(x, y) \mid x = -3k - 2, y = k, k \in Re\}$, line.

9. $\{(x, y) \mid x = -5k, y = k, k \in Re\}$, line.

10. $\{(x, y, z) \mid x = 7 - 2k, y = k, z = 2, k \in Re\}$, line.

11. $\{(x, y, z) \mid x = 6 - 6k, y = 4 - k, z = k, k \in Re\}$, line.

13. $\{(x, y, z) \mid x = 3 - k + 4l, y = k, z = l, k, l \in Re\}$, plane.

17. $\{(x, y, z) \mid x = 9 - 2k, y = k, z = 4, k \in Re\}$, line.

18. $\{(x, y, z) \mid x = 14k - 2, y = 2 - 5k, k, k \in Re\}$, line.

21. $\left\{(x, y) \mid x = \frac{4+k}{3}, y = k, k \in Re\right\}$, line.

22. $\{(x, y, z) \mid x = \frac{1-k}{7}, y = \frac{24+4k}{7}, z = k, k \in Re\}$, line.

23. $\{(x, y, z) \mid x = \frac{5-k}{2}, y = \frac{3k-5}{2}, z = k, k \in Re\}$, line.

27. $\{(x, y, z) \mid x = \frac{8-5k}{11}, y = \frac{13+7k}{11}, z = k, k \in Re\}$, line.

EXERCISE 6.9 (page 193)

1. Yes. **2.** Yes. **3.** No. **4.** No. **5.** No. **6.** No. **7.** No. **8.** No. **9.** Yes. **10.** No.

REVIEW EXERCISE 6 (page 194)

1. (a) $\{(\frac{9}{4}, \frac{19}{4})\}$. (b) $\{(\frac{11}{13}, \frac{-18}{13})\}$. (c) ϕ.
(d) $\{(3k - 3, k) \mid k \in Re\}$. (e) $\{(\frac{61}{11}, \frac{-4}{11})\}$. (f) ϕ.
(g) $\{(\frac{-38}{7}, \frac{-6}{7}, -7)\}$. (h) ϕ. (i) $\left\{\left(\frac{k-1}{10}, \frac{7-2k}{5}, k\right) \middle| k \in Re\right\}$.
(j) $\{(-1, 0, -3)\}$.

2. (a) Dependent, $\left\{\left(\frac{k+7}{6}, k\right) \middle| k \in Re\right\}$.
(b) Dependent, $\{(6 + 3k - 2l, k, l) \mid k, l \in Re\}$.

(c) Dependent, $\left\{\left(\frac{24-5k}{20}, \frac{2+5k}{10}, k\right) \mid k \in Re\right\}$.

(d) Dependent, $\left\{\left(\frac{18-2k}{7}, \frac{15+17k}{7}, k\right) \mid k \in Re\right\}$. (e) Inconsistent, ϕ.

(f) Dependent, $\left\{\left(\frac{k-2}{3}, k, 0\right) \mid k \in Re\right\}$.

(g) Dependent, $\left\{\left(\frac{18+3k-l}{5}, \frac{1+k+8l}{5}, k, l\right) \mid k, l \in Re\right\}$.

(h) Dependent, $\left\{\left(\frac{4+7k-13l}{21}, k, \frac{9+11l}{7}, l\right) \mid k, l \in Re\right\}$.

3. (a) Dependent, $\left\{\left(\frac{-4k}{3}, k \mid k \in Re\right\}\right.$.

(b) Dependent, $\{(-3k-3l, k, l) \mid k, l \in Re\}$.

(c) Independent, $\{(0, 0)\}$. (d) Dependent, $\left\{\left(\frac{4k}{3}, k\right) \mid k \in Re\right\}$.

(e) Dependent, $\left\{\left(\frac{3k}{8}, \frac{k}{16}, k\right) \mid k \in Re\right\}$.

(f) Dependent, $\{(-2k, k, 0) \mid k \in Re\}$. (g) Independent, $\{(0, 0, 0)\}$.

(h) Dependent, $\left\{\left(\frac{8k}{7}, \frac{10k}{7}, k\right) \mid k \in Re\right\}$. (i) Independent, $\{(0, 0)\}$.

(j) Independent, $\{(0, 0, 0)\}$.

(k) Dependent, $\left\{\left(\frac{k}{2}, \frac{5k-4l}{2}, k, l\right) \mid k, l \in Re\right\}$.

(l) Dependent, $\left\{\left(\frac{3k-5l}{15}, \frac{k}{5}, k, l\right) \mid k, l \in Re\right\}$. **4.** No.

5. (a) $\begin{bmatrix} 1 & 1 & 7 \\ 3 & -1 & 2 \end{bmatrix}$. (b) $\begin{bmatrix} 5 & -2 & 7 \\ 3 & 4 & -3 \end{bmatrix}$.

(c) $\begin{bmatrix} 3 & -2 & 7 \\ 6 & -4 & 21 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & -3 & -3 \\ -2 & 6 & 6 \end{bmatrix}$.

(e) $\begin{bmatrix} 2 & 3 & 10 \\ 1 & -4 & 7 \end{bmatrix}$. (f) $\begin{bmatrix} 1 & 2 & 6 \\ 3 & 6 & 12 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & 3 & -2 & 6 \\ 2 & -1 & -1 & -3 \\ 3 & 2 & -4 & 10 \end{bmatrix}$. (h) $\begin{bmatrix} 1 & -2 & 1 & -4 \\ 3 & -4 & 2 & 3 \\ 4 & -6 & 3 & 6 \end{bmatrix}$.

(i) $\begin{bmatrix} 6 & -1 & -1 & -2 \\ 2 & 3 & 1 & 4 \\ 2 & -7 & -3 & -10 \end{bmatrix}$. (j) $\begin{bmatrix} 3 & 1 & -2 & 3 \\ 1 & 4 & 0 & -1 \\ 0 & 3 & -2 & 6 \end{bmatrix}$.

6. (a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & \frac{-4}{3} \\ 0 & 1 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 7 & 3 \\ 0 & 1 & \frac{2}{11} \\ 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. (h) $\begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$.

(i) $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{13}{11} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. (j) $\begin{bmatrix} 1 & -2 & -2 & -4 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 1 & \frac{9}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

7. (a) Independent. (b) Independent. (c) Dependent. (d) Dependent.
(e) Dependent. (f) Inconsistent. (g) Independent. (h) Dependent.
(i) Dependent. (j) Independent.

8. (c) $\{(2 + 3k, k) \mid k \in Re\}$, line.

(d) $\left\{\left(\frac{5k - 2}{10}, \frac{5k + 14}{10}, k\right) \;\middle|\; k \in Re\right\}$, line.

(e) $\{(2k - 6, k, 5) \mid k \in Re\}$, line.

(h) $\left\{\left(\frac{9 - 8k}{7}, \frac{8 - 11k}{7}, k\right) \;\middle|\; k \in Re\right\}$, line.

(i) $\left\{\left(\frac{-k}{7}, \frac{2k}{7}, k\right) \middle| k \in Re\right\}$, line.

Chapter 7

EXERCISE 7.1 (page 199)

1. (a) 3, −, 2. (b) 6, 0, 5. (c) 4, −, 6. (d) 1, 3, −.
(e) 5, −, −. (f) 4, −1, 5.

2. (a) 2×2. (b) 2×3. (c) 3×2. (d) 1×4.
(e) 4×1. (f) 3×3.

3. No. **4.** $a = 5, b = -1$. **5.** Never.

6. (a) 11. (b) −1. (c) −10. (d) −5.

7. (a) −15. (b) 10. (c) 11. (d) 10.

EXERCISE 7.2 (page 201)

1. (a) $\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 7 \\ -9 & 2 \end{bmatrix}$. (c) $\begin{bmatrix} 8 & 3 \\ 3 & 3 \end{bmatrix}$. (d) $\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$.

(e) $\begin{bmatrix} 4 & 7 \\ -9 & 2 \end{bmatrix}$. (f) $\begin{bmatrix} 8 & 3 \\ 3 & 3 \end{bmatrix}$. (g) $\begin{bmatrix} 8 & 6 \\ -4 & 4 \end{bmatrix}$. (h) $\begin{bmatrix} 8 & 6 \\ -4 & 4 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 7 & -8 \\ 4 & 3 \end{bmatrix}$. (b) $\begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix}$. (c) $\begin{bmatrix} 0 & 3 \\ 0 & 9 \end{bmatrix}$. (d) $\begin{bmatrix} 7 & -8 \\ 4 & 3 \end{bmatrix}$.
(e) $\begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix}$. (f) $\begin{bmatrix} 0 & 3 \\ 0 & 9 \end{bmatrix}$. (g) $\begin{bmatrix} 2 & -4 \\ 5 & 9 \end{bmatrix}$. (h) $\begin{bmatrix} 2 & -4 \\ 5 & 9 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 30 & -6 \\ -6 & 18 \end{bmatrix}$. (b) $\begin{bmatrix} -6 & 21 \\ -15 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} \frac{-5}{2} & 2 \\ \frac{1}{2} & 3 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 19 & -17 \\ 7 & 9 \end{bmatrix}$. (b) $\begin{bmatrix} -35 & 22 \\ 7 & 24 \end{bmatrix}$. (c) $\begin{bmatrix} -1 & -22 \\ 11 & -30 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 3 & 3 \\ -2 & 6 \end{bmatrix}$. (b) $\begin{bmatrix} 9 & -5 \\ 9 & 6 \end{bmatrix}$. (c) $\begin{bmatrix} 9 & -5 \\ 9 & 6 \end{bmatrix}$. (d) $\begin{bmatrix} 17 & -8 \\ 9 & 15 \end{bmatrix}$.
(e) $\begin{bmatrix} 19 & -9 \\ 20 & 7 \end{bmatrix}$. (f) $\begin{bmatrix} 1 & 1 \\ -6 & 16 \end{bmatrix}$. (g) $\begin{bmatrix} 18 & 12 \\ 1 & 2 \end{bmatrix}$. (h) $\begin{bmatrix} 17 & 10 \\ 10 & 12 \end{bmatrix}$.

6. (a) $3, 4, 6, -4$. (b) $\frac{-4}{3}, -2, \frac{-13}{3}, 3$. (c) $5, \frac{20}{3}, 10, -\frac{20}{3}$.

8. (a) $\begin{bmatrix} 0 & -2 & 7 \\ 8 & 4 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 8 & 0 \\ -4 & 0 \\ -1 & 7 \end{bmatrix}$. (c) $\begin{bmatrix} 5 & -2 \\ -1 & 3 \\ 6 & 0 \\ 7 & 1 \end{bmatrix}$.
(d) $\begin{bmatrix} 9 & 2 & 1 \\ -5 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$. (e) $\begin{bmatrix} 1 & 5 & 3 \\ 7 & -4 & 6 \\ 4 & 2 & -1 \end{bmatrix}$. (f) No sum.

9. (a) Yes. (b) Yes. (c) Yes.

EXERCISE 7.3 (page 205)

5. $-A = \begin{bmatrix} -1 & -7 \\ 3 & -5 \end{bmatrix}$; $-B = \begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix}$; $-C = \begin{bmatrix} -4 & 2 \\ -1 & -7 \end{bmatrix}$.

6. (a) $\begin{bmatrix} 13 & 25 \\ -18 & 26 \end{bmatrix}$. (b) $\begin{bmatrix} 8 & 26 \\ -30 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 11 & 35 \\ -30 & 16 \end{bmatrix}$.

9. Yes.

EXERCISE 7.4 (page 209)

1. (a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (b) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. (c) $\begin{bmatrix} 11 \\ 1 \end{bmatrix}$. (d) $\begin{bmatrix} -3 \\ -6 \end{bmatrix}$. (e) $\begin{bmatrix} 5 \\ 10 \end{bmatrix}$.
(f) $\begin{bmatrix} -9 \\ 3 \end{bmatrix}$. (g) $\begin{bmatrix} -16 \\ 3 \end{bmatrix}$. (h) $\begin{bmatrix} -3 \\ 8 \end{bmatrix}$.

2. (a) $(0, 0)$. (b) $(\frac{8}{7}, -\frac{5}{7})$. (c) $(\frac{5}{7}, \frac{10}{7})$. (d) $(\frac{10}{7}, -\frac{8}{7})$.

3. (a) $R = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$. (b) $S = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$. (c) $T = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$.
(d) $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (e) $V = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$. (f) $W = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

4. (a) $u = 5x - y$, $v = x + y$. (b) $u = 2x - 3y$, $v = x$. (c) $u = 5x + 5y$, $v = -2x + 3y$.
(d) $u = x - y$, $v = y$. (e) $u = x$, $v = y$. (f) $u = y$, $v = x$.

5. (a) $\begin{bmatrix} -5 \\ -1 \end{bmatrix}$. (b) $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$. (c) $\begin{bmatrix} 20 \\ 12 \end{bmatrix}$. (d) $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$. (e) $\begin{bmatrix} -15 \\ 25 \end{bmatrix}$. (f) $\begin{bmatrix} -3 \\ -6 \end{bmatrix}$.

6. (a) $(\frac{12}{11}, -\frac{19}{11})$. (b) $(-\frac{2}{5}, -\frac{8}{5})$. (c) $(\frac{1}{7}, -\frac{2}{7})$.
(d) $(4, -3)$. (e) $(\frac{1}{3}, \frac{1}{5})$. (f) $(-\frac{7}{2}, \frac{3}{2})$.

7. (a) $\begin{bmatrix} 22 \\ -6 \end{bmatrix}$. (b) $\begin{bmatrix} -\frac{1}{4} \\ \frac{9}{8} \end{bmatrix}$.

8. $T\begin{bmatrix} \frac{a + 2b}{11} \\ \frac{-4a + 3b}{11} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

9. $T\begin{bmatrix} \frac{a + b}{4} \\ \frac{-a + 3b}{8} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

10. $T\begin{bmatrix} \frac{5b - a}{7} \\ \frac{2a - 3b}{7} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

EXERCISE 7.5 (page 214)

1. All images lie on the line $v = 3u$.
2. All images lie on the line $v = 2u$.
3. All images lie on the line $v = -2u$.
4. All images lie on the line $v = -\frac{1}{3}u$.
5. All images lie on the line $v = 2u$.
6. All images lie on the line $v = 4u$.
7. All images lie on the line $v = -u$.
8. All images lie on the line $v = 0$.
9. All images lie on the line $u = 0$.
10. $\{(2k - 1, k) \mid k \in Re\}$.

11. $\{(4k - 14, k) \mid k \in Re\}$.

12. $\left\{\left(\frac{k + 15}{3}, k\right) \;\middle|\; k \in Re\right\}$.

13. $\left\{\left(\frac{k - 8}{3}, k\right) \;\middle|\; k \in Re\right\}$.

14. $\left\{\left(\frac{k + 2}{4}, k\right) \;\middle|\; k \in Re\right\}$.

15. (1) $x - 2y = k$, $(k, 3k)$.
(2) $\frac{1}{2}x + y = k$, $(k, 2k)$.
(3) $-x + 4y = k$, $(k, -2k)$.
(4) $3x - 2y = k$, $(k, -\frac{1}{3}k)$.
(5) $3x - y = k$, $(k, 2k)$.
(6) $x + y = k$, $(k, 4k)$.
(7) $-x + \frac{1}{3}y = k$, $(k, -k)$.
(8) $x + 2y = k$, $(k, 0)$.
(9) $4x - y = k$, $(0, k)$.

16. $ad - bc \neq 0$.

EXERCISE 7.6 (page 218)

F, R, U, W do not have inverses.

1. $x = -\frac{1}{8}u + \frac{3}{8}v$, $y = \frac{3}{8}u - \frac{1}{8}v$.
2. $x = \frac{1}{5}u + \frac{2}{5}v$, $y = \frac{1}{5}u - \frac{3}{5}v$.
3. $x = \frac{2}{5}u - \frac{1}{5}v$, $y = \frac{1}{5}u - \frac{3}{5}v$.

4. $x = \frac{u}{3}$, $y = \frac{v}{2}$. **5.** $x = \frac{v}{2}$, $y = \frac{u}{3}$. **8.** $x = \frac{v}{3}$, $y = u - \frac{v}{3}$.

9. $x = \frac{3}{4}u + \frac{1}{2}v$, $y = \frac{1}{2}u$. **11.** $x = \frac{1}{5}u + \frac{1}{5}v$, $y = \frac{2}{5}u - \frac{3}{5}v$.

13. (a) $(7, 5)$, $(7, -1)$, $(1, -3)$, $(3, 4)$, $(6, 2)$, $(3, 3)$, $(4, -4)$, $(5, 0)$.
(b) $(-3, -9)$, $(-9, -3)$, $(-9, -3)$, $(-9, 0)$, $(0, -6)$, $(-3, -9)$, $(0, -6)$, $(-9, -6)$.
(c) $(-13, 1)$, $(-4, 7)$, $(11, 12)$, $(6, -10)$, $(-15, 4)$, $(-3, 6)$, $(-10, 19)$, $(1, 9)$.
(d) $(-10, 2)$, $(-2, 6)$, $(10, 10)$, $(6, -8)$, $(-12, 4)$, $(-2, 6)$, $(-8, 16)$, $(2, 8)$.

EXERCISE 7.7 (page 221)

1. (a) $\begin{bmatrix} -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -19 \\ -5 \end{bmatrix}$. (b) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \end{bmatrix}$. (c) $\begin{bmatrix} 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -27 \\ 15 \end{bmatrix}$.
(d) $\begin{bmatrix} -2 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 13 \\ -4 \end{bmatrix}, \begin{bmatrix} 43 \\ 9 \end{bmatrix}$. (b) $\begin{bmatrix} -5 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 15 \end{bmatrix}$. (c) $\begin{bmatrix} -8 \\ 16 \end{bmatrix}, \begin{bmatrix} 72 \\ -40 \end{bmatrix}$.
(d) $\begin{bmatrix} 3 \\ 17 \end{bmatrix}, \begin{bmatrix} -11 \\ 33 \end{bmatrix}$.

3. (a) $r = 14x - 5y$, $s = 6x + y$. (b) $r = 8x + 7y$, $s = -5y$. (c) $r = 9x - 18y$, $s = -5x + 10y$.
(d) $r = 5x + 7y$, $s = -15x - 21y$.

4. $r = 10x - 3y$, $s = 5x - 7y$.

5. $A^{-1}\begin{cases} x = \frac{3}{11}u + \frac{2}{11}v \\ y = \frac{-1}{11}u + \frac{3}{11}v \end{cases}$; $B^{-1}\begin{cases} u = \frac{1}{5}r + \frac{1}{5}s \\ v = \frac{2}{5}r - \frac{3}{5}s \end{cases}$.

6. $(BA)^{-1}\begin{cases} x = \frac{7}{55}r - \frac{3}{55}s \\ y = \frac{1}{11}r - \frac{2}{11}s \end{cases}$.

7. $A^{-1}B^{-1}\begin{cases} x = \frac{7}{55}r - \frac{3}{55}s \\ y = \frac{1}{11}r - \frac{2}{11}s \end{cases}$.

8. $BA\begin{cases} r = 6x \\ s = 7x + y \end{cases}$; $A^{-1}\begin{cases} x = \frac{v}{2} \\ y = u - \frac{5}{2}v \end{cases}$; $B^{-1}\begin{cases} u = -\frac{1}{3}r + s \\ v = \frac{1}{3}r \end{cases}$;
$(BA)^{-1}\begin{cases} x = \frac{1}{6}r \\ y = -\frac{7}{6}r + s \end{cases}$; $A^{-1}B^{-1} = (BA)^{-1}$.

REVIEW EXERCISE 7 (page 222)

1. (a) $-1, -1, 3$. (b) 18.

2. (a) $\begin{bmatrix} 7 & -1 \\ 4 & 5 \end{bmatrix}$. (b) $\begin{bmatrix} -4 & 0 \\ 1 & -3 \end{bmatrix}$. (c) $\begin{bmatrix} 17 & -3 \\ 13 & 12 \end{bmatrix}$.

4. $\begin{bmatrix} 4 & -5 \\ 5 & 6 \end{bmatrix}$.

5. (a) $\begin{aligned} r &= -x + 3y \\ s &= 5x + 2y \end{aligned}$. (b) $\begin{bmatrix} 8 \\ -23 \end{bmatrix}$. (c) $\begin{bmatrix} \frac{1}{17}\frac{3}{7} \\ \frac{-24}{17} \end{bmatrix}$.

(d) $T\begin{bmatrix} \frac{3}{17}n - \frac{2}{17}m \\ \frac{n}{17} + \frac{5}{17}m \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$.

6. (a) All image points lie on the line $v = -3u$.

(b) Line $-x + 4y = k \leftrightarrow$ point $(k, -3k)$.

7. (a) $\begin{aligned} x &= \tfrac{1}{6}u + \tfrac{1}{6}v \\ y &= \tfrac{1}{4}u - \tfrac{1}{4}v \end{aligned}$. (b) $\begin{aligned} x &= -\tfrac{1}{3}u + \tfrac{1}{3}v \\ y &= 2u - v \end{aligned}$. (c) None.

8. (a) $\begin{aligned} r &= 12x + y \\ s &= 15x + 5y \end{aligned}$. (b) $\begin{aligned} w &= 24x + 2y \\ z &= 45x + 15y \end{aligned}$. (c) $\begin{aligned} w &= 6u + 4v \\ z &= 18u + 3v \end{aligned}$.

(d) $\begin{aligned} w &= 24x + 2y \\ z &= 45x + 15y \end{aligned}$.

Chapter 8

EXERCISE 8.1 (page 226)

1. $\begin{bmatrix} -10 & 7 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} -15 & -2 \\ 25 & 7 \end{bmatrix}$.

2. $\begin{bmatrix} 18 & 9 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 13 & 19 \\ 2 & 5 \end{bmatrix}$.

3. $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$.

4. $\begin{bmatrix} 2 & -8 \\ 11 & 26 \end{bmatrix}, \begin{bmatrix} -1 & -13 \\ 13 & 29 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

EXERCISE 8.2 (page 231)

1. (a) $\begin{bmatrix} 4 & -2 \\ 23 & -9 \end{bmatrix}$. (b) $\begin{bmatrix} -7 & -2 \\ 12 & 2 \end{bmatrix}$. (c) $\begin{bmatrix} 2 & 12 \\ -6 & 14 \end{bmatrix}$.

(d) $\begin{bmatrix} 14 & -5 \\ -25 & 9 \end{bmatrix}$. (e) $\begin{bmatrix} 56 & -10 \\ 82 & -20 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 22 & 7 \\ 26 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} -7 & 7 \\ -3 & 29 \end{bmatrix}$. (c) $\begin{bmatrix} 11 & 15 \\ 6 & 26 \end{bmatrix}$.

(d) $\begin{bmatrix} 19 & 115 \\ 46 & 134 \end{bmatrix}$. (e) $\begin{bmatrix} 211 & 555 \\ 222 & 766 \end{bmatrix}$.

3. (a) $\begin{bmatrix} -2 & 5 \\ 4 & 16 \end{bmatrix}$. (b) $\begin{bmatrix} -5 & 5 \\ 19 & 7 \end{bmatrix}$. (c) $\begin{bmatrix} 3 & -1 \\ 4 & 12 \end{bmatrix}$.

(d) $\begin{bmatrix} 11 & 6 \\ 1 & 33 \end{bmatrix}$. (e) $\begin{bmatrix} 6 & -4 \\ -19 & -3 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} -1 & -14 \\ 35 & 6 \end{bmatrix}$. (e) $\begin{bmatrix} -1 & -14 \\ 35 & 6 \end{bmatrix}$.

5. (a) $\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}$. (b) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 4 & -1 \\ 5 & -1 \end{bmatrix}$. (e) $\begin{bmatrix} 5 & -1 \\ 4 & -1 \end{bmatrix}$.

6. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA$.

7. (a) $\begin{bmatrix} 16 \\ 34 \end{bmatrix}$. (b) $[-2 \quad 23]$. (c) Not possible. (d) Not possible.

(e) $\begin{bmatrix} 0 \\ 31 \end{bmatrix}$. (f) $[-3]$. (g) $\begin{bmatrix} 9 & -8 \\ 34 & -7 \end{bmatrix}$. (h) $[-14 \quad -6]$.

(i) $\begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$. (j) $\begin{bmatrix} -5 & 1 & -5 \\ 6 & 12 & 4 \\ -2 & 8 & -4 \end{bmatrix}$.

8. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B^2$; $C^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

If $a^2 = b^2$, $a, b \in Re$, then $a = \pm b$. But $A \neq B$.

If $a^2 = 0$, $a \in Re$, then $a = 0$. But $C \neq 0$.

9. (a) AB yes, BA no. (b) AB yes, BA no. (c) Yes. (d) Yes. (e) No.

10. BA if $n = r$; AB if $s = m$.

EXERCISE 8.3 (page 236)

1. (a) and (b) $\begin{bmatrix} -12 & -10 \\ 39 & 15 \end{bmatrix}$; (c) and (d) $\begin{bmatrix} -7 & 0 \\ 28 & 0 \end{bmatrix}$.

2. (a) and (b) $\begin{bmatrix} -2 & -6 \\ 34 & 62 \end{bmatrix}$; (c) and (d) $\begin{bmatrix} -6 & 4 \\ 32 & 12 \end{bmatrix}$.

3. (a) and (b) $\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}$; (c) and (d) $\begin{bmatrix} 11 & 6 \\ 7 & 5 \end{bmatrix}$.

4. (a) and (b) $\begin{bmatrix} 88 & 16 \\ -2 & 28 \end{bmatrix}$; (c) and (d) $\begin{bmatrix} 20 & 36 \\ 33 & -11 \end{bmatrix}$.

5. $A(AB) = A^2B = \begin{bmatrix} 51 & -13 \\ -34 & 71 \end{bmatrix}$; $A(A + B) = A^2 + AB = \begin{bmatrix} 24 & 0 \\ -16 & 34 \end{bmatrix}$.

6. $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. **7.** $B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$. **8.** $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

EXERCISE 8.4 (page 241)

1. $\begin{bmatrix} \frac{4}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{4}{15} \end{bmatrix}$. **2.** $\begin{bmatrix} 0 & \frac{1}{5} \\ 1 & \frac{3}{5} \end{bmatrix}$. **3.** $\begin{bmatrix} \frac{3}{8} & -\frac{2}{8} \\ \frac{1}{8} & \frac{2}{8} \end{bmatrix}$. **4.** $\begin{bmatrix} \frac{2}{9} & \frac{5}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{bmatrix}$.

5. No inverse. **6.** $\begin{bmatrix} \frac{1}{8} & -\frac{3}{40} \\ \frac{3}{8} & -\frac{1}{40} \end{bmatrix}$. **7.** $\begin{bmatrix} -\frac{1}{8} & -\frac{3}{40} \\ -\frac{3}{8} & -\frac{1}{40} \end{bmatrix}$. **8.** No inverse.

9. $\begin{bmatrix} 0 & -1 \\ \frac{1}{2} & 1 \end{bmatrix}$. **10.** $\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{4}{3} & 1 \end{bmatrix}$. **11.** $\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$. **12.** $\begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{4} & 0 \end{bmatrix}$.

13. No inverse. **14.** No inverse. **15.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. **16.** $\begin{bmatrix} \frac{6}{23} & \frac{-3}{23} \\ \frac{-2}{23} & \frac{24}{23} \end{bmatrix}$.

17. $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$. **18.** $\begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$.

19. (a) $\begin{bmatrix} 2 & 6 \\ -10 & 12 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{1}{7} & -\frac{1}{14} \\ \frac{5}{42} & \frac{1}{42} \end{bmatrix}$. (c) $\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$.

(d) $\begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{14} & \frac{3}{14} \end{bmatrix}$.

EXERCISE 8.5 (page 244)

1. (a) $\begin{bmatrix} 17 \\ 22 \end{bmatrix}$. (b) $\begin{bmatrix} 31 \\ 16 \end{bmatrix}$. (c) $\begin{bmatrix} 41 \\ -2 \end{bmatrix}$. (d) $\begin{bmatrix} 12 \\ 15 \end{bmatrix}$.

(e) $\begin{bmatrix} -2 \\ 13 \end{bmatrix}$. (f) $\begin{bmatrix} -5 \\ -4 \end{bmatrix}$. (g) $\begin{bmatrix} 5a + 3b \\ -a + 4b \end{bmatrix}$. (h) $\begin{bmatrix} 3x + 2y \\ -2x + 3y \end{bmatrix}$.

(i) $\begin{bmatrix} 4s \\ -s + 3t \end{bmatrix}$. (j) $\begin{bmatrix} 6m - n \\ 4m + 5n \end{bmatrix}$.

2. (a) $\begin{bmatrix} 6 & -1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. (b) $\begin{bmatrix} 2 & 5 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

(c) $\begin{bmatrix} 3 & 2 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. (d) $\begin{bmatrix} 2 & -7 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$.

(e) $\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (f) $\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

3. (a) -2. (b) No value. (c) -3.

4. $WA = [0\ 19]$; AW is not defined.

EXERCISE 8.6 (page 247)

1. $\{(\frac{28}{13}, \frac{9}{13})\}$. **2.** $\{(\frac{13}{7}, \frac{10}{7})\}$. **3.** $\{(\frac{5}{9}, \frac{26}{9})\}$. **4.** Inconsistent.

5. $\{(\frac{31}{24}, \frac{-75}{24})\}$. **6.** $\{(-1, 4)\}$. **7.** $\{(\frac{-8}{17}, \frac{22}{17})\}$. **8.** $\{(\frac{-4}{13}, \frac{-12}{13})\}$.

9. $\{(\frac{3}{5}, \frac{8}{5})\}$. **10.** $\left\{\left(\frac{2m + 3n}{23}, \frac{5m - 4n}{23}\right)\right\}$.

EXERCISE 8.7 (page 251)

1. $\begin{bmatrix} -\frac{7}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{1}{8} \end{bmatrix}$. **2.** $\begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$. **3.** No inverse. **4.** $\begin{bmatrix} \frac{2}{9} & \frac{1}{3} \\ -\frac{1}{18} & \frac{1}{6} \end{bmatrix}$.

5. $\begin{bmatrix} \frac{1}{6} & \frac{1}{2} \\ -\frac{1}{12} & \frac{1}{4} \end{bmatrix}$. **6.** No inverse. **7.** $\begin{bmatrix} \frac{8}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{8}{5} \end{bmatrix}$. **8.** $\begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$.

9. $\begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

EXERCISE 8.8 (page 254)

1. (a) V_2', V_2'. (b) V_2', V_2'. (c) V_2', V_2'.

(d) V_2', $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \in Re, a = b \right\}$. (e) V_2', V_2'. (f) V_2', V_2'.

(g) V_2', V_2'. (h) V_2', $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \in Re, b = -3a \right\}$.

EXERCISE 8.9 (page 261)

1. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. (c) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. (e) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (f) $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$.

(g) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. (h) $\begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

3. $(-\sqrt{3}, 1)$, $(-1 + \frac{\sqrt{3}}{2}, -\sqrt{3} - \frac{1}{2})$, $(1 + \frac{\sqrt{3}}{2}, \sqrt{3} - \frac{1}{2})$.

4. $(0, 4)$, $(-4, -2)$, $(4, -2)$.

5. (a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. (b) $(2, 2)$, $(6, 2)$, $(10, 7)$.
(c) $(-3, 2)$, $(-9, 2)$, $(-15, 7)$.

6. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. (b) $(1, -9)$, $(1, 15)$, $(2, 24)$.
(c) $(1, 3)$, $(1, -5)$, $(2, -8)$.

7. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

8. (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. (b) $(-1, 3)$, $(-2, 5)$, $(-3, 7)$.

(c) $(1, -3), (2, -5), (3, -7)$. (d) $(-1, -3), (-2, -5), (-3, -7)$.

9. (a) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$. (b) $(0, 0), (3, 0), (13, 4)$. (c) $(0, 0), (3, -6), (1, 2)$.

10. If the magnification in the y-direction is $u = x + 0y, v = 0x + ky$, the required transformation is $u = x + 0y, v = 0x - ky$.

11. $\begin{bmatrix} k & ka \\ 0 & k \end{bmatrix}$, k from the uniform magnification and a from the shear.

12. $\begin{bmatrix} -1 & -ka \\ 0 & -k \end{bmatrix}$, k from the uniform magnification and a from the shear.

REVIEW EXERCISE 8 (page 263)

1. $BA = \begin{bmatrix} 20 & 8 \\ -18 & -2 \end{bmatrix}$. **2.** $AB = \begin{bmatrix} 14 & 2 \\ -24 & 4 \end{bmatrix}$.

4. (a) $\begin{bmatrix} -3 & 12 \\ 7 & -7 \end{bmatrix}$. (b) $\begin{bmatrix} -6 & -15 \\ 4 & 3 \end{bmatrix}$. (c) $\begin{bmatrix} 6 & 6 \\ 3 & -6 \end{bmatrix}$. (d) $\begin{bmatrix} 29 & -5 \\ -13 & 21 \end{bmatrix}$.

5. (a) No. (b) Yes. (c) No. (d) Yes.

7. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; A and B are divisors of zero. **8.** $\begin{bmatrix} \frac{2}{9} & \frac{1}{27} \\ -\frac{1}{9} & \frac{4}{27} \end{bmatrix}$.

9. (a) $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{15} & \frac{2}{15} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{1}{7} & \frac{5}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$. (c) $\begin{bmatrix} 7 & -7 \\ 4 & 11 \end{bmatrix}$. (d) $\begin{bmatrix} \frac{11}{105} & \frac{1}{15} \\ \frac{-4}{105} & \frac{1}{15} \end{bmatrix}$.

10. (a) $\begin{bmatrix} 16 & 4 \\ -8 & 3 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{3}{80} & \frac{-1}{20} \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix}$. (c) $\begin{bmatrix} \frac{1}{5} & 0 \\ \frac{1}{5} & 1 \end{bmatrix}$.

11. (a) $\begin{bmatrix} 40 \\ 10 \end{bmatrix}$; $\begin{bmatrix} 20 \\ -1 \end{bmatrix}$; $\begin{bmatrix} 20 \\ 15 \end{bmatrix}$. **12.** $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

14. (a) $\{(\frac{-2}{13}, \frac{8}{13})\}$. (b) $\{(\frac{5}{9}, \frac{-7}{9})\}$. (c) $\{(\frac{4}{3}, 2)\}$.

15. (a) V_2', V_2'. (b) $V_2', \left\{\begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in Re, b = -a\right\}$.

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